VL Formal Modeling (WS 2022)

Symbolic Summation and the modeling of sequences

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Most of the ideas presented below (in particular starting fro Section 2) are taken from

> C. Schneider Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation. In: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computuation , pp. 423-485. 2021. Springer, ISBN 978-3-030-80218-9. arXiv:2102.01471 [cs.SC].

More precisely, a special case is extracted and simplified to make it digestible by a compact series of lectures. Relevant literature of the presented material can be found in the above article.

Contents

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1 Symbolic Summation (a short introduction)

In the following, we will give some basic ideas how identities can be discovered in a formal setting. We start with the following telescoping problem:

Given an expression $f(k)$ that evaluates to a sequence.
Find an expression $g(k)$ such that the telescoping equation

$$f(k) = g(k+1) - g(k) \tag{1}$$

holds.

Suppose we find such an expression $g(k)$. Then summing (1) over k from a to b (and assuming that no poles arise during the evaluation) yields

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a).$$

We note that we could always choose

$$g(k) = \sum_{i=a}^{k-1} f(i). \tag{2}$$

Thus we should refine our problem from above:

Find an expression $g(k)$ with (1) where $g(k)$ is simpler than the trivial solution (2).

Later we will form a term algebra in which we can define our sequences properly and will introduce the formal setting of difference rings to model them accordingly

1.1 Indefinite summation of polynomials

We start with one of the most simplest cases: the summand is a polynomial, i.e., $f(x) \in \mathbb{K}[x]$. The following questions arise:

1. What is the domain of expressions in which we search $g(k)$?

2. How can we calculate a solution $g(k)$ in this solution domain?

As it turns out, the first question can be answered nicely: a solution $g(x)$ exists always in $\mathbb{K}[x]$. For the second question, we will consider two different tactics that are often used in summation packages.

• *Tactic 1: the classical approach.* In the following it will be convenient to use also the difference operator. For any sequence (expression) $g(k)$ we write

$$\Delta g(k) = g(k+1) - g(k).$$

Note that for indefinite integration of polynomials one can utilize the following well known property: for any $m \in \mathbb{N}$ we have

$$D_x x^m = m\, x^{m-1} \tag{3}$$

and hence

$$\int_a^b x^m dx = \frac{x^{m+1}}{m+1}\Big|_a^b = \frac{b^{m+1} - a^{m+1}}{m+1}. \tag{4}$$

Thus by linearity we can integrate any polynomial by

$$\int_a^b \sum_{m=0}^{d} c_m\, x^m dx = \sum_{m=0}^{d} c_m \int_a^b x^m dx = \sum_{m=0}^{d} \frac{c_m(b^{m+1} - a^{m+1})}{m+1}. \tag{5}$$

For indefinite summation of polynomials we can follow precisely the same classical strategy. We start with some simple definitions.

Definition 1.1. *For a function $p : \mathbb{N} \to \mathbb{K}$ we define the difference operator*

$$\Delta p(x) = p(x+1) - p(x).$$

Below we will apply Δ to polynomials $p(x) \in \mathbb{K}[x]$ and rational functions $p(x) \in \mathbb{K}(x)$.

Definition 1.2. *Let R be a commutative ring R with 1. For $a \in R$ and $m \in \mathbb{N}$ we define the* falling factorials *of a w.r.t. m by*

$$a^{\underline{m}} = \begin{cases} 1 & \text{if } m = 0 \\ a(x-1)\dots(a-m+1) & \text{if } m \geq 1. \end{cases}$$

Example 1.3. For $p = \frac{1}{x} \in \mathbb{Q}(x)$ we have

$$\Delta(p) = p(x+1) - p(x) = \frac{1}{x+1} - \frac{1}{x} = -\frac{1}{x(1+x)}.$$

Further for $a = x \in \mathbb{Q}[x]$ we have

$$a^{\underline{3}} = x(x-1)(x-2) \in \mathbb{Q}[x].$$

Integration of polynomials can be carried over to summation of polynomials by utilizing the following property.

Lemma 1.4. *For $m \in \mathbb{N}$ we have*

$$\boxed{\Delta x^{\underline{m}} = m\, x^{\underline{m-1}}}.$$

Proof. We have

$$\begin{aligned}\Delta x^{\underline{m}} &= (x+1)^{\underline{m}} - x^{\underline{m}}\\ &= (x+1)x(x-1)\dots(x-m+2) - x(x-1)\dots(x-m+1)\\ &= ((x+1)-(x-m+1))x(x-1)\dots(x-m+2)\\ &= m\, x^{\underline{m-1}}.\end{aligned}$$

□

As a consequence we get

$$x^{\underline{m}} = \Delta \frac{x^{\underline{m+1}}}{m+1}, \quad m \in \mathbb{N}$$

and summing this equation over k from a to b yields

$$\sum_{x=a}^{b} x^{\underline{m}} = \frac{(b+1)^{\underline{m+1}} - a^{\underline{m+1}}}{m+1}.$$

Note that this is nothing else than the discrete version given in (3) and (4). In particular, for given

$$f(x) = \sum_{m=0}^{d} c_m\, x^{\underline{m}} \in \mathbb{K}[x] \tag{6}$$

with $d \in \mathbb{N}$ it follows that

$$g(x) = \sum_{m=0}^{d} \frac{c_m x^{\underline{m+1}}}{m+1}$$

is a telescoping solution of (1). Furthermore, analogously to (5) we obtain

$$\sum_{k=a}^{b} f(x) = \sum_{m=0}^{d} c_m \sum_{k=a}^{b} k^{\underline{m}} = \sum_{m=0}^{d} \frac{c_m((b+1)^{\underline{m+1}} - a^{\underline{m+1}})}{m+1}.$$

The only problem is that in many cases one does not have a polynomial given in the representation (6) for some $d \in \mathbb{N}$ but in the form

$$\sum_{m=0}^{d} \bar{c}_m\, x^m \in \mathbb{K}[x].$$

Luckily one can rewrite a polynomial written in the basis

$$1, x, x^2, \dots, x^d$$

to the representation written in the basis

$$x^{\underline{0}} = 1, x^{\underline{1}} = x, x^{\underline{2}} = x(x-1), \ldots, x^{\underline{d}} = x(x-1)\ldots(x-d+1)$$

by using the formula

$$x^m = \sum_{k=0}^{m} S(m,k) x^{\underline{k}}$$

where $S(m,k)$ denotes the Stirling numbers of second kind. They can be computed by

$$S(m,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^m;$$

alternatively, there is the recurrence formula

$$S(m+1,k) = k\, S(m,k) + S(m,k-1)$$

with the initial values $S(0,0) = 1$ and $S(m,0) = S(0,m) = 0$ for $m \geq 1$.

Example 1.5. Consider the polynomial

$$f(x) = x^4.$$

Using the formulas from above, we get

$$f(x) = x^4 = \sum_{k=0}^{4} S(4,k) x^{\underline{k}} = 0x^{\underline{0}} + 1x^{\underline{1}} + 7x^{\underline{2}} + 6x^{\underline{3}} + 1x^{\underline{4}}.$$

Thus we get

$$\begin{aligned} g(x) &= \frac{1}{2}x^{\underline{2}} + \frac{7}{3}x^{\underline{3}} + \frac{3}{2}x^{\underline{4}} + \frac{1}{5}x^{\underline{5}} \\ &= \frac{1}{30}(x-1)x(2x-1)(3x^2-3x-1). \end{aligned}$$

such that

$$g(x+1) - g(x) = f(x)$$

holds. In particular we get

$$\sum_{k=1}^{n} k^4 = \sum_{k=1}^{n} f(k) = g(n+1) - g(1) = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1). \tag{7}$$

• *Tactic 2: linear algebra.* We use the following property: for $f(x) \in \mathbb{K}[x]$ there is a $g(x) \in \mathbb{K}[x]$ with (1) where

$$\deg(g) \leq \deg(f) + 1.$$

Thus the desired solution has the form

$$g(x) = \sum_{m=0}^{d} g_m\, x^m$$

and we can determine the unknowns $g_0, \ldots, g_d \in \mathbb{K}$ by linear algebra.

Example 1.6. Take $f(x) = x^4 \in \mathbb{Q}[x]$. Then we can make the ansatz

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + g_4 x^4 + g_5 x^5$$

for the unknowns $g_0, g_1, g_2, g_3, g_4, g_5 \in \mathbb{Q}$. This gives

$$\begin{aligned} x^4 =& \Delta g(x) = g(x+1) - g(x) \\ &= 0x^5 \\ &+ 5g_5 x^4 \\ &+ (4g_4 + 10g_5)x^3 \\ &+ (3g_3 + 6g_4 + 10g_5)x^2 \\ &+ (2g_2 + 3g_3 + 4g_4 + 5g_5)x \\ &+ (g_1 + g_2 + g_3 + g_4 + g_5)x^0. \end{aligned}$$

By coefficient comparison this yields the linear system

$$\begin{aligned} &[x^4] & 1 =& 5g_5 \\ &[x^3] & 0 =& 4g_4 + 10g_5 \\ &[x^2] & 0 =& 3g_3 + 6g_4 + 10g_5 \\ &[x^1] & 0 =& 2g_2 + 3g_3 + 4g_4 + 5g_5 \\ &[x^0] & 0 =& g_1 + g_2 + g_3 + g_4 + g_5 \end{aligned}$$

which is already in triangular form. Thus we can read off the solution

$$g_5 = \frac{1}{5}, \qquad g_4 = -\frac{1}{2}, \qquad g_3 = \frac{1}{3}, \qquad g_2 = 0, \qquad g_1 = -\frac{1}{30}, \qquad g_0 = c$$

with $c \in \mathbb{Q}$. In particular, we can choose $c = 0$ and obtain

$$g(x) = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30} = \frac{1}{30}(x-1)x(2x-1)\left(3x^2 - 3x - 1\right).$$

To this end, we continue as in the previous example and get (7).

1.2 Rational summation and beyond

Example 1.7. With telescoping algorithms we can show that there is no $g(x) \in \mathbb{K}(x)$ such that

$$g(x+1) - g(x) = \frac{1}{x}$$

holds. This shows that there is no rational function $h(x) \in \mathbb{K}(x)$ such that

$$h(n) = \sum_{k=1}^{n} \frac{1}{k}$$

holds for all $n \geq \delta$ for some $\delta \in \mathbb{N}$. We call this special sequence also the Harmonic numbers and denote it by

$$H_n := \sum_{k=1}^{n} \frac{1}{k}.$$

Example 1.8. Take

$$f(x) = \frac{x^2 + x + 1}{(x+1)(x+2)}.$$

Then telescoping algorithms show that there is not $g(x) \in \mathbb{Q}(x)$ with

$$g(x+1) - g(x) = f(x).$$

But using, e.g., the summation package `Sigma` we find

$$g(x) = \frac{2x^2 + 7x - 1}{2(x+1)} - 2H_x$$

which yields

$$\sum_{k=1}^{n} f(k) = g(n+1) - g(1) = \frac{(2n+3)(n+5)n}{2(n+1)(n+2)} - 2H_n.$$

With the summation package

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-JKU

this simplification can be accomplished as follows. One can insert the above sum

In[2]:= **mySum = SigmaSum[(k² + k + 1)/(k + 1)/(k + 2), {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n} \frac{k^2 + k + 1}{(k+1)(k+2)}$

and can apply the command

In[3]:= **SigmaReduce[mySum]**

Out[3]= $\frac{(2n+3)(n+5)n}{2(n+1)(n+2)} - 2H_n$

This indicates that we have to search in larger domains than $\mathbb{K}(x)$.

Example 1.9. Given

$$f(x) = H_x.$$

Compute (if it exists) a $g(x) \in$ "$\mathbb{K}(x)[H_x]$" with

$$g(x+1) - g(x) = H_x.$$

Note that $H_{x+1} = H_x + \frac{1}{x+1}$.
Similarly to polynomial summation we have the following structural property (provided by summation theory):
If there is a solution $g(x)$ with the shape specified above then

$$\deg(g) = \deg(f) + 1 = 2.$$

Thus, if there is such a solution, then

$$g(x) = g_0 + g_1 H_x + g_2 H_x^2$$

with $g_0, g_1, g_2 \in \mathbb{K}(x)$. By linear system solving (plus solving variants of the telescoping problem in $\mathbb{K}(x)$) we obtain the solution

$$g_0 = -x, \quad g_1 = x, \quad g_2 = 0,$$

i.e.,

$$g(x) = xH_x - x = x(H_x - 1)$$

This yields

$$\begin{aligned}\sum_{k=0}^{n} H_k &= g(n+1) - \underbrace{g(0)}_{=0} \\ &= (n+1)(H_{n+1} - 1) = -n + (1+n)H_n.\end{aligned}$$

Using the package `Sigma`, we define our sum

In[4]:= **mySum = SigmaSum[SigmaHNumber[k], {k, 0, n}]**

Out[4]= $\sum_{k=0}^{n} H_k$

and simplify it automatically (as indicated above) by executing

In[5]:= **SigmaReduce[mySum]**

Out[5]= $-n + (1 + n)H_n$

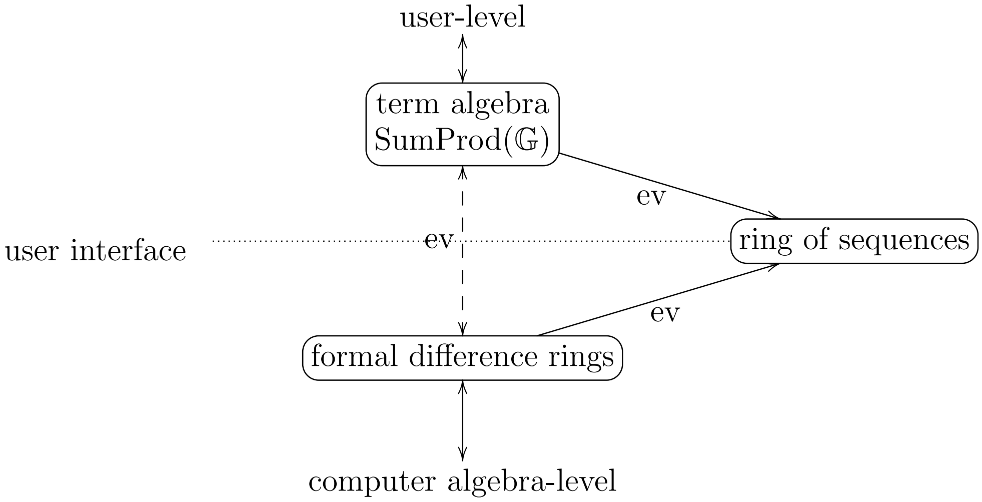
The last example was rather sloppy: We treated the harmonic numbers not as a sequence but as a variable. In particular, we considered $\mathbb{K}(x)[H_x]$ as a polynomial ring in the variable H_x.

Sequences have to be handled accordingly in a formal setting (i.e., in formal rings generated by variables).

An appropriate model will be considered in more details in the next lecture for the general class of indefinite nested sums defined over hypergeometric products.

2 Modeling of sequences with a term algebra - the user interface

In order to represent indefinite nested sums defined over hypergeometric products we will deal with the following interplay of different modeling levels.



In this section we will focus on the concepts above the user interface.
Set $\mathbb{G} = \mathbb{K}(x)$. For any element $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x]$ where $q \neq 0$ and p, q being coprime we define

$$\operatorname{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0. \end{cases} \tag{8}$$

Note that there is a $\delta \in \mathbb{N}$ with $q(k) \neq 0$ for all $k \in \mathbb{N}$ with $k \geq \delta$. We define $L(f)$ to be the minimal value $\delta \in \mathbb{N}$ such that $q(k) \neq 0$ holds for all $k \geq \delta$; further, we define $Z(f) = \max(L(1/p), L(1/q))$ for $f \neq 0$. In short, we have introduced the functions $\operatorname{ev} : \mathbb{G} \times \mathbb{N} \to \mathbb{K}$ and $L, Z : \mathbb{G} \to \mathbb{N}$.

Example 2.1. Take $f = \frac{p}{q} \in \mathbb{G}$ with $p = x - 4$ and $q = (x-3)(x-1) \in \mathbb{K}[x]$. Note that in the evaluation

$$(\operatorname{ev}(f, n))_{n \geq 0} = (-\tfrac{4}{3}, \underline{0}, 2, \underline{0}, 0, \tfrac{1}{8}, \dots) \in \mathbb{Q}^{\mathbb{N}}$$

the underlined 0 are coming from the arising poles. In particular, starting from $n \geq L(f)) = 4$ such pole evaluations (leading to zero) do not arise. Furthermore, starting from $n \geq Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) = \max(4, 5) = 5$ no zeroes arise within the evaluation.

Now we extend $\mathbb{G}$ to expressions SumProd($\mathbb{G}$) in terms of indefinite nested sums defined over hypergeometric products. A special treatment will be done for the set

$$\mathcal{R} = \{r \in \mathbb{K} \setminus \{1\} \mid r \text{ is a root of unity}\};$$

here we introduce the function $\mathrm{ord} : \mathcal{R} \to \mathbb{Z}_{\geq 1}$ with

$$\mathrm{ord}(r) = \min\{n \in \mathbb{Z}_{\geq 1} \mid r^n = 1\}.$$

Let $\owedge$, $\oplus$, $\odot$, Sum, Prod and RPow be operations with the signatures

$$\begin{array}{lll}
\owedge : & \mathrm{SumProd}(\mathbb{G}) \times \mathbb{Z} & \to \mathrm{SumProd}(\mathbb{G}) \\
\oplus : & \mathrm{SumProd}(\mathbb{G}) \times \mathrm{SumProd}(\mathbb{G}) & \to \mathrm{SumProd}(\mathbb{G}) \\
\odot : & \mathrm{SumProd}(\mathbb{G}) \times \mathrm{SumProd}(\mathbb{G}) & \to \mathrm{SumProd}(\mathbb{G}) \\
\mathrm{Sum} : & \mathbb{N} \times \mathrm{SumProd}(\mathbb{G}) & \to \mathrm{SumProd}(\mathbb{G}) \\
\mathrm{Prod} : & \mathbb{N} \times \mathrm{SumProd}(\mathbb{G}) & \to \mathrm{SumProd}(\mathbb{G}) \\
\mathrm{RPow} : & \mathcal{R} & \to \mathrm{SumProd}(\mathbb{G}).
\end{array}$$

In the following we write $\owedge$, $\oplus$ and $\odot$ in infix notation, and Sum and Prod in prefix notation. Further, for $(\dots((f_1 \square f_2) \square f_3) \square \dots \square f_r)$ with $\square \in \{\odot, \oplus\}$ and $f_1, \dots, f_r \in \mathrm{SumProd}(\mathbb{G})$ we write $f_1 \square f_2 \square f_3 \square \dots \square f_r$.
More precisely, we define the following chain of set inclusions:

$$\mathrm{Prod}^*(\mathbb{G}) \subset \mathrm{SumProd}(\mathbb{G})$$

as follows. $\mathrm{Prod}^*(\mathbb{G})$ which is the smallest set that contains 1 with the following properties:

1. If $r \in \mathcal{R}$ then $\mathrm{RPow}(r) \in \mathrm{Prod}^*(\mathbb{G})$.
2. If $f \in \mathbb{G}^*$ and $l \in \mathbb{N}$ with $l \geq Z(f)$ then $\mathrm{Prod}(l, f) \in \mathrm{Prod}^*(\mathbb{G})$.
3. If $p, q \in \mathrm{Prod}^*(\mathbb{G})$ then $p \odot q \in \mathrm{Prod}^*(\mathbb{G})$.
4. If $p \in \mathrm{Prod}^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p \owedge z \in \mathrm{Prod}^*(\mathbb{G})$.

Furthermore, we define

$$\Pi(\mathbb{G}) = \{\mathrm{RPow}(r) \mid r \in \mathcal{R}\} \cup \{\mathrm{Prod}(l, f) \mid f \in \mathbb{G}, l \in \mathbb{N}\} \subset \Pi(\mathbb{G}).$$

Example 2.2. In $\mathbb{G} = \mathbb{Q}(x)$ we get

$$P = (\underbrace{\mathrm{Prod}(1, x)}_{\in \Pi(\mathbb{G})} \owedge (-2)) \odot \underbrace{\mathrm{RPow}(-1)}_{\Pi(\mathbb{G})} \in \mathrm{Prod}^*(\mathbb{G}).$$

Finally, we define *SumProd*$(\mathbb{G})$ as the smallest set containing $\mathbb{G} \cup \mathrm{Prod}^*(\mathbb{G})$ with the following properties:

1. For all $f, g \in \mathrm{SumProd}(\mathbb{G})$ we have $f \oplus g \in \mathrm{SumProd}(\mathbb{G})$.
2. For all $f, g \in \mathrm{SumProd}(\mathbb{G})$ we have $f \odot g \in \mathrm{SumProd}(\mathbb{G})$.
3. For all $f \in \mathrm{SumProd}(\mathbb{G})$ and $k \in \mathbb{Z}_{\geq 1}$ we have $f \owedge k \in \mathrm{SumProd}(\mathbb{G})$.
4. For all $f \in \mathrm{SumProd}(\mathbb{G})$ and $l \in \mathbb{N}$ we have $\mathrm{Sum}(l, f) \in \mathrm{SumProd}(\mathbb{G})$.

Furthermore, we introduce the *set of nested sums over hypergeometric products* given by

$$\Sigma(\mathbb{G}) = \{\mathrm{Sum}(l, f) \mid l \in \mathbb{N} \text{ and } f \in \mathrm{SumProd}(\mathbb{G})\}$$

and the *set of nested sums and hypergeometric products* given by

$$\Sigma\Pi(\mathbb{G}) = \Sigma(\mathbb{G}) \cup \Pi(\mathbb{G}).$$

Example 2.3. With $\mathbb{G} = \mathbb{K}(x)$ we get, e.g., the following expressions:

$$\begin{aligned}
&E_1 = \mathrm{Sum}(1, \mathrm{Prod}(1, x)) \in \Sigma(\mathbb{G}) \subset \mathrm{SumProd}(\mathbb{G}),\\
&E_2 = \mathrm{Sum}(1, \tfrac{1}{x+1} \odot \mathrm{Sum}(1, \tfrac{1}{x^3}) \odot \mathrm{Sum}(1, \tfrac{1}{x})) \in \Sigma(\mathbb{G}) \subset \mathrm{SumProd}(\mathbb{G}),\\
&E_3 = (E_1 \oplus E_2) \odot E_1 \in \mathrm{SumProd}(\mathbb{G}).
\end{aligned}$$

Finally, we introduce a function ev (a model of the term algebra) which evaluates a given expression of our term algebra to sequence elements. Here we start with the evaluation function $\mathrm{ev} : \mathbb{G} \times \mathbb{N} \to \mathbb{K}$ given by (8) and extend it recursively from $\mathbb{G}$ to $\mathrm{ev}\colon \mathrm{SumProd}(\mathbb{G}) \times \mathbb{N} \to \mathrm{SumProd}(\mathbb{G})$ as follows.

1. For $f, g \in \mathrm{SumProd}(\mathbb{G})$ and $k \in \mathbb{Z} \setminus \{0\}$ ($k > 0$ if $f \notin \mathrm{Prod}^*(\mathbb{G})$) we set

$$\begin{aligned}
\mathrm{ev}(f \circledwedge k, n) &:= \mathrm{ev}(f, n)^k,\\
\mathrm{ev}(f \oplus g, n) &:= \mathrm{ev}(f, n) + \mathrm{ev}(g, n),\\
\mathrm{ev}(f \odot g, n) &:= \mathrm{ev}(f, n)\ \mathrm{ev}(g, n);
\end{aligned}$$

2. for $r \in \mathcal{R}$ and $\mathrm{Sum}(l, f), \mathrm{Prod}(\lambda, g) \in \mathrm{SumProd}(\mathbb{G})$ we define

$$\begin{aligned}
\mathrm{ev}(\mathrm{RPow}(r), n) &:= \prod_{i=1}^{n} r = r^n,\\
\mathrm{ev}(\mathrm{Sum}(l, f), n) &:= \sum_{i=l}^{n} \mathrm{ev}(f, i),\\
\mathrm{ev}(\mathrm{Prod}(\lambda, g), n) &:= \prod_{i=\lambda}^{n} \mathrm{ev}(g, i) = \prod_{i=\lambda}^{n} g(i).
\end{aligned}$$

Remark 2.4. (0) $\prod_{i=\lambda} g(i)$ for $g(x) \in \mathbb{K}(x)$ with $\lambda \geq Z(g)$ is called hypergeometric product.
(1) Since $\mathrm{ev}(\mathrm{Prod}(r, l), n) = \mathrm{ev}(\mathrm{RPow}(r), n)$, RPow is redundant. But it will be convenient for the treatment of canonical representations (see Definition 2.11).
(2) Any evaluation of $\mathrm{Prod}^*(\mathbb{G})$ is well defined and nonzero since the lower bounds of the products are set large enough via the z-function.

ev applied to $f \in \mathrm{SumProd}(\mathbb{G})$ represents a sequence. In particular, f can be considered as a simple program and $\mathrm{ev}(f, n)$ with $n \in \mathbb{N}$ executes it (like an interpreter/compiler) yielding the nth entry of the represented sequence.

Definition 2.5. *For $F \in SumProd(\mathbb{G})$ and $n \in \mathbb{N}$ we write $F(n) := \mathrm{ev}(F, n)$.*

Example 2.6. For $E_i \in \mathrm{SumProd}(\mathbb{K}(x))$ with $i = 1, 2, 3$ in Ex. 2.3 we get

$$E_1(n) = \mathrm{ev}(E_1, n) = \sum_{k=1}^{n} \prod_{i=1}^{k} i = \sum_{k=1}^{n} k!, \qquad E_2(n) = \mathrm{ev}(E_2, n) = \sum_{k=1}^{n} \tfrac{1}{1+k} \Big(\sum_{i=1}^{k} \tfrac{1}{i^3} \Big) \sum_{i=1}^{k} \tfrac{1}{i}$$

and $E_3(n) = (E_1(n) + E_2(n))E_1(n)$. For $P \in \mathrm{SumProd}(\mathbb{K}(x))$ in Ex. 2.2 we get

$$P(n) = \mathrm{ev}(P, n) = \Big(\prod_{k=1}^{n} k \Big)^{-2} (-1)^n.$$

Example 2.7. We show how the expressions of $\mathrm{SumProd}(\mathbb{G})$ with ev are handled in

In[6]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-JKU

Instead of $F = \mathrm{Sum}(1, \frac{1}{x})$ with $F(n) = \mathrm{ev}(F, n) = \sum_{k=1}^{n} \frac{1}{k}$ we introduce the sum by

In[7]:= $\mathbf{F = SigmaSum[\frac{1}{k}, \{k, 1, n\}]}$

Out[7]= $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$

where n is kept symbolically. However, if the user replaces n by a concrete integer, say 5, the evaluation mechanism is carried out and we get $F(5) = \mathrm{ev}(F, 5)$:

In[8]:= $\mathbf{F/.n \to 5}$

Out[8]= $\displaystyle\frac{137}{60}$

Similarly, we can define E_1 from Example 2.3 as follows:

In[9]:= $\mathbf{E_1 = SigmaSum[SigmaFactorial[k], \{k, 1, n\}]}$

Out[9]= $\displaystyle\sum_{k=1}^{n} k!$

Here `SigmaFactorial` defines the factorials; its full definition is given by:

In[10]:= $\mathbf{GetFullDefinition[E_1]}$

Out[10]= $\displaystyle\sum_{k=1}^{n} \prod_{o_1=1}^{k} o_1$

Similarly, one can introduce as shortcuts powers, Pochhammer symbols, binomials , harmonic numbers or more generally harmonic sums with the function calls `SigmaPower`, `SigmaPochhammer`, `SigmaBinomial` or `SigmaHNumber` or `S`.

In the same fashion, we can define $E_2, E_3 \in \mathrm{SumProd}(\mathbb{Q}(x))$ from Example 2.3 and $P \in \mathrm{SumProd}(\mathbb{Q}(q)(x, x_1))$ with $q = q_1$ from Example 2.2 by

In[11]:= $\mathbf{E_2 = SigmaSum[SigmaSum[1/i, \{i, 1, k\}]SigmaSum[1/i^3, \{i, 1, k\}]/(k + 1), \{k, 1, n\}]}$

Out[11]= $\displaystyle\sum_{k=1}^{n} \frac{\Big(\sum_{i=1}^{k} \frac{1}{i^3}\Big) \sum_{i=1}^{k} \frac{1}{i}}{1+k}$

In[12]:= $\mathbf{E_3 = (E_1 + E_2)E_1}$

Out[12]= $\left(\sum_{k=1}^{n} k!\right)\left(\sum_{k=1}^{n} k! + \sum_{k=1}^{n} \frac{\left(\sum_{i=1}^{k} \frac{1}{i^3}\right) \sum_{i=1}^{k} \frac{1}{i}}{1+k}\right)$

In[13]:= **P = SigmaProduct[k, {k, 1, n}]$^{-2}$SigmaPower[−1, n]**

Out[13]= $(-1)^{n}\left(\prod_{k=1}^{n} \frac{1}{k}\right)^{2}$

Note that within `Sigma` the root of unity product $\mathrm{RPow}(\alpha)$ with $\alpha \in \mathcal{R}$ can be either defined by `SigmaPower[`α`,n]` or `SigmaProduct[`$\alpha, \{k, 1, n\}$`]`. Whenever α is recognized as an element of $\mathcal{R}$, it is treated as the special product $\mathrm{RPow}(\alpha)$.

Expressions in $\mathrm{SumProd}(\mathbb{G})$ (similarly within Mathematica using `Sigma`) can be written in different ways such that they produce the same sequence. In the remaining part of this section we will elaborate on canonical (unique) representations.

In a preprocessing step we can rewrite the expressions to a reduced representation.

Definition 2.8. *An expression $A \in SumProd(\mathbb{G})$ is in* reduced representation *if*

$$A = (f_1 \odot P_1) \oplus (f_2 \odot P_2) \oplus \dots \oplus (f_r \odot P_r) \tag{9}$$

with $f_i \in \mathbb{G}^*$ and

$$P_i = (a_{i,1} \circledwedge z_{i,1}) \odot (a_{i,2} \circledwedge z_{i,2}) \odot \dots \odot (a_{i,n_i} \circledwedge z_{i,n_i}) \tag{10}$$

for $1 \leq i \leq r$ where

- *$a_{i,j} = Sum(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in SumProd(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z}_{\geq 1}$,*
- *$a_{i,j} = Prod(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in Prod^*(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z} \setminus \{0\}$, or*
- *$a_{i,j} = RPow(f_{i,j})$ with $f_{i,j} \in \mathcal{R}$ and $1 \leq z_{i,j} < ord(r_{i,j})$*

such that the following properties hold:

1. *for each $1 \leq i \leq r$ and $1 \leq j < j' < n_i$ we have $a_{i,j} \neq a_{i,j'}$;*
2. *for each $1 \leq i < i' \leq r$ with $n_i = n_j$ there does not exist a $\sigma \in S_{n_i}$ with $P_{i'} = (a_{i,\sigma(1)} \circledwedge z_{i,\sigma(1)}) \odot (a_{i,\sigma(2)} \circledwedge z_{i,\sigma(2)}) \odot \dots \odot (a_{i,\sigma(n_i)} \circledwedge z_{i,\sigma(n_i)})$.*

We say that $H \in SumProd(\mathbb{G})$ is in sum-product reduced representation *(or in* sum-product reduced form*) if it is in reduced representation and for each $Sum(l, A)$ and $Prod(l, A)$ that occur recursively in H the following holds: A is in reduced representation as given in* (9)*, $l \geq \max(L(f_1), \dots, L(f_r))$ (i.e. the first case of* (8) *is avoided during evaluations) and the lower bound l is greater than or equal to the lower bounds of the sums and products inside of A.*

Lemma 2.9. *For any $A \in SumProd(\mathbb{G})$, there is a $B \in SumProd(\mathbb{G})$ in sum-product reduced representation and $\lambda \in \mathbb{N}$ such that $A(n) = B(n)$ holds for all $n \geq \lambda$.*

Example 2.10. In `Sigma` the reduced representation of E_3 is calculated with the call

In[14]:= **CollectProdSum[E3, 3]**

Out[14]= $$\Big(\sum_{k=1}^{n} k!\Big)^2 + \Big(\sum_{k=1}^{n} k!\Big) \sum_{k=1}^{n} \frac{\Big(\sum_{i=1}^{k} \frac{1}{i^3}\Big) \sum_{i=1}^{k} \frac{1}{i}}{1+k}$$

Before we can state one of `Sigma`'s crucial features we need the following definitions.

Definition 2.11. *Let $W \subseteq \Sigma\Pi(\mathbb{G})$. We define* $\mathrm{SumProd}(W, \mathbb{G})$ *as the set of elements from SumProd($\mathbb{G}$) which are in reduced representation and where the arising sums and products are taken from W. More precisely, $A \in$ SumProd($W, \mathbb{G}$) if and only if it is of the form* (9) *with* (10) *where $a_{i,j} \in W$. In the following we seek a W with the following properties:*

- *W is called* shift-closed over $\mathbb{G}$ *if for any $A \in$ SumProd($W, \mathbb{G}$), $s \in \mathbb{Z}$ there are $B \in$ SumProd($W, \mathbb{G}$) and $\delta \in \mathbb{N}$ such that $A(n+s) = B(n)$ holds for all $n \geq \delta$.*
- *W is called* shift-stable over $\mathbb{G}$ *if for any product or sum in W the multiplicand or summand is built by sums and products from W.*
- *W is called* canonical reduced over $\mathbb{G}$ *if for any $A, B \in$ SumProd($W, \mathbb{G}$) with $A(n) = B(n)$ for all $n \geq \delta$ for some $\delta \in \mathbb{N}$ the following holds: A and B are the same up to permutations of the operands in $\oplus$ and $\odot$.*

Note: If W is shift-stable, than it is shift-closed. But the other direction does not hold.

Example 2.12. The set

$$W = \{\underbrace{\mathrm{Sum}(1, \tfrac{1}{x})}_{=A_1}, \underbrace{\mathrm{Sum}(1, \tfrac{1}{x} \odot \mathrm{Sum}(1, \tfrac{1}{x}))}_{A_2}, \underbrace{\mathrm{Sum}(1, \tfrac{1}{x} \odot \mathrm{Sum}(1, \tfrac{1}{x^2}))}_{A_3}\}$$

is not shift-stable since $\mathrm{Sum}(1, \frac{1}{x}^2)$ which occurs in A_3 is not in the set W. But it is shift-closed. E.g., we have

$$\begin{aligned} A_1(n+1) &= A_1(n) + \frac{1}{n+1} \\ A_2(n+1) &= A_2(n) + \frac{A_1(n) + \frac{1}{n+1}}{n+1} \\ A_3(n+1) &= A_3(n) + \frac{\sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2}}{n+1}. \end{aligned}$$

In particular, we have

$$\sum_{k=1}^{n} \frac{1}{k^2} = 2A_2(n) - A_1(n)^2$$

Thus with

$$\begin{aligned} B_1 &= A_1 \oplus \frac{1}{x+1} \\ B_2 &= A_2 \oplus (\frac{1}{x+1} \odot A_1) \oplus \frac{1}{(x+1)^2} \\ B_3 &= A_3 \oplus \left(\frac{2}{x+1} \odot A_2\right) \oplus \left(\frac{-1}{x+1} \odot A_1 \circledwedge 2\right) \oplus \frac{1}{(x+1)^3} \end{aligned}$$

we have $A_i(n+1) = B_i(n)$ for all $n \geq 0$.

Based on this observation, we focus on σ-reduced sets which we define as follows.

Definition 2.13. $W \subseteq \Sigma\Pi(\mathbb{G})$ *is called* σ-reduced over $\mathbb{G}$ *if*

1. *the elements in* W *are in sum-product reduced form,*
2. W *is shift-stable and*
3. W *is canonical reduced.*

In particular, $A \in \mathit{SumProd}(W, \mathbb{G})$ *is called* σ-reduced (w.r.t. W) *if* W *is* σ*-reduced over* $\mathbb{G}$.

More precisely, we are interested in the following problem.

Problem SigmaReduce: Compute a σ-reduced representation

Given: $A_1, \dots, A_u \in \mathrm{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.
Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$, $B_1 \dots, B_u \in \mathrm{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that for all $1 \leq i \leq r$ we get
$$A_i(n) = B_i(n) \quad n \geq \delta_i.$$

Example 2.14. Consider the following two expressions from $\mathrm{SumProd}(\mathbb{Q}(x))$:

In[15]:= $\mathbf{A_1 = SigmaSum[SigmaSum[1/i, \{i, 1, k\}]SigmaSum[1/i^3, \{i, 1, k\}]/(k + 1), \{k, 1, n\}]}$

Out[15]= $\sum_{k=1}^{n} \frac{\left(\sum_{i=1}^{k} \frac{1}{i^3}\right) \sum_{i=1}^{k} \frac{1}{i}}{1+k}$

In[16]:= $\mathbf{A_2} = \sum_{i=1}^{n} \frac{1}{i^5} - \frac{\sum_{i=1}^{n} \frac{1}{i^4}}{1+n} - \sum_{j=1}^{n} \frac{\sum_{i=1}^{j} \frac{1}{i^4}}{j} - \sum_{j=1}^{n} \frac{\sum_{i=1}^{j} \frac{1}{i^3}}{j^2} + \frac{\sum_{j=1}^{n} \frac{\sum_{i=1}^{j} \frac{1}{i^3}}{j}}{1+n} - \sum_{j=1}^{n} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j^4} + \frac{\sum_{j=1}^{n} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j^3}}{1+n} + \sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i^3}}{j}}{k} + \sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j^3}}{k};$

Then we solve Problem SigmaReduce by executing:

In[17]:= $\mathbf{\{B_1, B_2\} = SigmaReduce[\{A_1, A_2\}, n]}$

Out[17]= $\{\sum_{k=1}^{n} \frac{\left(\sum_{i=1}^{k} \frac{1}{i^3}\right) \sum_{i=1}^{k} \frac{1}{i}}{1+k}, \sum_{k=1}^{n} \frac{\left(\sum_{i=1}^{k} \frac{1}{i^3}\right) \sum_{i=1}^{k} \frac{1}{i}}{1+k}\}$

Since $B_1 = B_2$, it follows $A_1 = A_2$. Note that the set W pops up only implicitly. The set of all sums and products in the output, in our case

$$W_0 = \Big\{ \sum_{k=1}^{n} \frac{1}{1+k} \Big(\sum_{i=1}^{k} \frac{1}{i^3} \Big) \sum_{i=1}^{k} \frac{1}{i} \Big\} (= \{\mathrm{Sum}(1, \tfrac{1}{x+1} \odot \mathrm{Sum}(1, \tfrac{1}{x^3}) \odot \mathrm{Sum}(1, \tfrac{1}{x}))\})$$

forms a canonical set in which A_1 and A_2 can be represented by B_1 and B_2 respectively. Adjoining in addition all sums and products that arise inside of the elements in W_0 we get $W = \{\sum_{i=1}^{n} \frac{1}{i}, \sum_{i=1}^{n} \frac{1}{i^3}\} \cup W_0$ which is a σ-reduced set. Internally, `SigmaReduce` parses the arising objects from left to right and constructs the underlying σ-reduced set W in which the input expressions can be rephrased.

Reversing the order of the input elements yields the following result:

In[18]:= $\mathbf{\{B_2, B_1\} = SigmaReduce[\{A_2, A_1\}, n]}$

Out[18]= $$\Big\{-\Big(\sum_{k=1}^{n}\frac{1}{k^4}\Big)\sum_{k=1}^{n}\frac{1}{k}+\frac{\Big(\sum_{k=1}^{n}\frac{1}{k^3}\Big)\sum_{k=1}^{n}\frac{1}{k}}{1+n}-\sum_{k=1}^{n}\frac{\sum_{k=1}^{k}\frac{1}{k^3}}{k^2}+\sum_{k=1}^{n}\frac{\sum_{k=1}^{k}\frac{1}{k^4}}{k}+\sum_{k=1}^{n}\frac{\Big(\sum_{k=1}^{k}\frac{1}{k^3}\Big)\sum_{k=1}^{k}\frac{1}{k}}{k},$$
$$-\Big(\sum_{k=1}^{n}\frac{1}{k^4}\Big)\sum_{k=1}^{n}\frac{1}{k}+\frac{\Big(\sum_{k=1}^{n}\frac{1}{k^3}\Big)\sum_{k=1}^{n}\frac{1}{k}}{1+n}-\sum_{k=1}^{n}\frac{\sum_{k=1}^{k}\frac{1}{k^3}}{k^2}+\sum_{k=1}^{n}\frac{\sum_{k=1}^{k}\frac{1}{k^4}}{k}+\sum_{k=1}^{n}\frac{\Big(\sum_{k=1}^{k}\frac{1}{k^3}\Big)\sum_{k=1}^{k}\frac{1}{k}}{k}\Big\}$$

In this case we get the σ-reduced set

$$W=\Big\{\sum_{j=1}^{n}\frac{1}{j^4},\sum_{j=1}^{n}\frac{1}{j^3},\sum_{j=1}^{n}\frac{1}{j},\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^4}}{j},\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j^2},\sum_{j=1}^{n}\frac{\Big(\sum_{k=1}^{j}\frac{1}{k^3}\Big)\sum_{k=1}^{j}\frac{1}{k}}{j}\Big\}$$

(expressed in the `Sigma`-language) and since $B_1 = B_2$ we conclude again that $A_1 = A_2$ holds for all $n \geq 0$. To check that $A_1 = A_2$ holds, one can also execute

In[19]:= $\mathbf{SigmaReduce[A_1 - A_2, n]}$

Out[19]= 0

Here $W = \{\}$ is the σ-reduced set in which we can represent $A_1 - A_2$ by 0.
In order to extract the full information (with all the relations of the arising sums), one can also take each sum separately

In[20]:= $\mathbf{varL = Complement[Variables[\{A_1, A_2\}], \{n\}];}$

and can compute their σ-reduced representations with

In[21]:= **varLRed = SigmaReduce[varL, n];**

In this case we get with

In[22]:= **substL = MapThread[Rule, {varL, varLRed}];**

the following set of substitution rules:

$$\sum_{i=1}^{n}\frac{1}{i^5}\to\sum_{j=1}^{n}\frac{1}{j^5},\quad \sum_{i=1}^{n}\frac{1}{i^4}\to\sum_{j=1}^{n}\frac{1}{j^4},\quad \sum_{j=1}^{n}\frac{\sum_{i=1}^{j}\frac{1}{i^4}}{j}\to\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^4}}{j},$$
$$\sum_{j=1}^{n}\frac{\sum_{i=1}^{j}\frac{1}{i^3}}{j^2}\to\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j^2},\quad \sum_{j=1}^{n}\frac{\sum_{i=1}^{j}\frac{1}{i^3}}{j}\to\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j},$$
$$\sum_{j=1}^{n}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j^4}\to\sum_{j=1}^{n}\frac{1}{j^5}+\Big(\sum_{j=1}^{n}\frac{1}{j^4}\Big)\sum_{j=1}^{n}\frac{1}{j}-\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^4}}{j},$$
$$\sum_{j=1}^{n}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j^3}\to\sum_{j=1}^{n}\frac{1}{j^4}+\Big(\sum_{j=1}^{n}\frac{1}{j^3}\Big)\sum_{j=1}^{n}\frac{1}{j}-\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j},$$
$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i^3}}{j}}{k}\to\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j^2}+\Big(\sum_{j=1}^{n}\frac{1}{j}\Big)\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j}-\sum_{j=1}^{n}\frac{\Big(\sum_{k=1}^{j}\frac{1}{k^3}\Big)\sum_{k=1}^{j}\frac{1}{k}}{j}, \tag{11}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j^3}}{k}\to\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^4}}{j}-\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j^2}-\sum_{j=1}^{n}\frac{1}{j}\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j}+2\sum_{j=1}^{n}\frac{\left(\sum_{k=1}^{j}\frac{1}{k^3}\right)\sum_{k=1}^{j}\frac{1}{k}}{j},$$

$$\sum_{k=1}^{n}\frac{\left(\sum_{i=1}^{k}\frac{1}{i^3}\right)\sum_{i=1}^{k}\frac{1}{i}}{1+k}\to\left(-\sum_{j=1}^{n}\frac{1}{j^4}+\frac{\sum_{j=1}^{n}\frac{1}{j^3}}{1+n}\right)\sum_{j=1}^{n}\frac{1}{j}+\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^4}}{j}-\sum_{j=1}^{n}\frac{\sum_{k=1}^{j}\frac{1}{k^3}}{j^2}+\sum_{j=1}^{n}\frac{\left(\sum_{k=1}^{j}\frac{1}{k^3}\right)\sum_{k=1}^{j}\frac{1}{k}}{j}.$$

Note that here the left-hand sides equal the right-hand sides.

Such a unique representation (up to trivial permutations) immediately gives rise to the following application: One can compare if two expressions A_1 and A_2 evaluate to the same sequences (from a certain point on): simply check if the resulting B_1 and B_2 in $\mathrm{SumProd}(W, \mathbb{G})$ for a σ-reduced W are the same (up to trivial permutations). Alternatively, just check if $A_1 - A_2$ can be reduced to zero.

As it turns out, the theory of difference rings provides all the techniques necessary to tackle the above problems. In the next section we introduce all the needed ingredients and will present our main result in Theorem 3.19 below.

November 9, 2022

3 Modeling sequences in difference rings - computer algebra

In the following we will rephrase expressions $H \in \mathrm{SumProd}(\mathbb{G})$ as elements h in a formal difference ring. More precisely, we will design

- a ring $\mathbb{A}$ with $\mathbb{A} \supseteq \mathbb{G} \supseteq \mathbb{K}$ in which H can be represented by $h \in \mathbb{A}$;
- an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ such that $H(n) = \mathrm{ev}(h, n)$ holds for sufficiently large $n \in \mathbb{N}$;
- a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ which models the shift $H(n+1)$ with $\sigma(h)$.

Example 3.1. We will rephrase $F = \mathrm{Sum}(1, \frac{1}{x}) \in \mathrm{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$ where $\mathbb{K} = \mathbb{Q}$ in a formal ring. Namely, we take the polynomial ring $\mathbb{A} = \mathbb{G}[s] = \mathbb{Q}(x)[s]$ (s transcendental over $\mathbb{G}$) and extend $\mathrm{ev} : \mathbb{G} \times \mathbb{N} \to \mathbb{Q}$ to $\mathrm{ev}' : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$ as follows: for $h = \sum_{k=0}^{d} f_k \, s^k$ with $f_k \in \mathbb{G}$ we set

$$\mathrm{ev}'(h, n) := \sum_{k=0}^{d} \mathrm{ev}(f_k, n) \, \mathrm{ev}'(s, n)^k \tag{12}$$

with

$$\mathrm{ev}'(s, n) = \sum_{i=1}^{n} \frac{1}{i} =: S_1(n) (= H_n); \tag{13}$$

since ev and ev′ agree on $\mathbb{G}$, we do not distinguish them anymore. For any

$$H = f_0 \oplus (f_1 \odot (F \circledwedge 1)) \oplus \cdots \oplus (f_d \odot (F \circledwedge d))$$

with $d \in \mathbb{N}$ and $f_0, \ldots, f_d \in \mathbb{G}$ we can take $h = \sum_{k=0}^{d} f_k s^k \in \mathbb{A}$ and get

$$H(n) = \mathrm{ev}(h, n) \quad \forall n \in \mathbb{N}.$$

Further, we introduce the shift operator acting on the elements in $\mathbb{A}$. For the field $\mathbb{G}$ we simply define the field automorphism $\sigma : \mathbb{G} \to \mathbb{G}$ with $\sigma(f) = f|_{x \mapsto x+1} (= f(x+1))$. Moreover, based on the observation that for any $n \in \mathbb{N}$ we have

$$F(n+1) = \sum_{i=1}^{n+1} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i} + \frac{1}{n+1},$$

we extend the automorphism $\sigma : \mathbb{G} \to \mathbb{G}$ to $\sigma' : \mathbb{A} \to \mathbb{A}$ as follows: for $h = \sum_{k=0}^{d} f_k \, s^k$ with $f_k \in \mathbb{G}$ we set $\sigma'(h) := \sum_{k=0}^{d} \sigma(f_k) \sigma'(s)^k$ with $\sigma'(s) = s + \frac{1}{x}$; since σ' and σ agree on $\mathbb{G}$, we do not distinguish them anymore. We observe that

$$\mathrm{ev}(s, n+1) = \sum_{i=1}^{n+1} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i} + \frac{1}{n+1} = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = \mathrm{ev}(\sigma(s), n)$$

holds for all $n \in \mathbb{N}$ and more generally that $\mathrm{ev}(h, n+l) = \mathrm{ev}(\sigma^l(h), n)$ holds for all $h \in \mathbb{A}$, $l \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $n \geq \max(-l, 0)$.

As illustrated in the example above, the following definitions will be relevant.

Definition 3.2. *A* difference ring *is a ring $\mathbb{A}$ equipped with a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ which one also denotes by $(\mathbb{A}, \sigma)$.*
For a difference ring $(\mathbb{A}, \sigma)$ and a subfield $\mathbb{K}$ of $\mathbb{A}$ with $\sigma|_{\mathbb{K}} = \mathrm{id}$ we introduce the following functions.

1. *A function* $\mathrm{ev}\colon \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ *is called* evaluation function *for $(\mathbb{A}, \sigma)$ if for all $f, g \in \mathbb{A}$ and $c \in \mathbb{K}$ there exists a $\lambda \in \mathbb{N}$ with the following properties:*
$$\forall n \geq \lambda : \mathrm{ev}(c, n) = c, \tag{14}$$
$$\forall n \geq \lambda : \mathrm{ev}(f + g, n) = \mathrm{ev}(f, n) + \mathrm{ev}(g, n), \tag{15}$$
$$\forall n \geq \lambda : \mathrm{ev}(f\, g, n) = \mathrm{ev}(f, n)\ \mathrm{ev}(g, n). \tag{16}$$
In addition, we require that for all $f \in \mathbb{A}$ and $l \in \mathbb{Z}$ there exists a λ with
$$\forall n \geq \lambda : \mathrm{ev}(\sigma^l(f), n) = \mathrm{ev}(f, n + l). \tag{17}$$

2. *A function $L\colon \mathbb{A} \to \mathbb{N}$ is called an* operation-function *(in short o-*function*) for $(\mathbb{A}, \sigma)$ and an evaluation function* ev *if for any $f, g \in \mathbb{A}$ with $\lambda = \max(L(f), L(g))$ the properties* (15) *and* (16) *hold and for any $f \in \mathbb{A}$ and $l \in \mathbb{Z}$ with $\lambda = L(f) + \max(0, -l)$ property* (17) *holds.*

We note that a construction of a map $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ with the properties (14) and (16) is straightforward. It is property (17) that brings in extra complications: the evaluation of the elements in $\mathbb{A}$ must be compatible with the automorphism σ.

Example 3.3. Take the rational function field $\mathbb{G} := \mathbb{G} = \mathbb{K}(x)$ with the function (8), together with the functions $L : \mathbb{G} \to \mathbb{N}$ and $Z\colon \mathbb{G}^* \to \mathbb{N}$ from the beginning of Section 2. It is easy to see that $\mathrm{ev} : \mathbb{G} \times \mathbb{N} \to \mathbb{K}$ satisfies for all $c \in \mathbb{K}$ and $f, g \in \mathbb{G}$ the property (14) for $L(c) = 0$ and the properties (15) and (16) with $\lambda = \max(L(f), L(g))$. Finally, we take the automorphism $\sigma\colon \mathbb{G} \to \mathbb{G}$ defined by $\sigma|_{\mathbb{K}} = \mathrm{id}$ and $\sigma(x) = x + 1$. Then one can verify in addition that (17) holds for all $f \in \mathbb{G}$ and $l \in \mathbb{Z}$ with $\lambda = \max(-l, L(f))$. Consequently, ev is an evaluation function for $(\mathbb{G}, \sigma)$ and L is an o-function for $(\mathbb{G}, \sigma)$. In addition, Z is a z-function for ev and $\mathbb{G}^*$ by construction. In the following we call $(\mathbb{G}, \sigma)$ also the *rational difference field.*

In the following we look for such a formal difference ring $(\mathbb{A}, \sigma)$ with a computable evaluation function ev and o-function L in which we can model a finite set of expressions $A_1, \dots, A_u \in \mathrm{SumProd}(\mathbb{G})$ with $a_1, \dots, a_u \in \mathbb{A}$.

Definition 3.4. *Let $F \in \mathit{SumProd}(\mathbb{G})$ and $(\mathbb{A}, \sigma)$ be a difference ring extension of $(\mathbb{G}, \sigma)$ equipped with an evaluation function* $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$*. We say that $f \in \mathbb{A}$* models *F if $\mathrm{ev}(f, n) = F(n)$ holds for all $n \geq \lambda$ for some $\lambda \in \mathbb{N}$.*

3.1 The naive representation in APS-extensions

As indicated in Example 3.1 our sum-product expressions will be rephrased in a tower of difference ring extensions.

Definition 3.5. *A difference ring* $(\mathbb{E},\sigma)$ *is called an* APS-extension *of a difference ring* $(\mathbb{A},\sigma)$ *if* $\mathbb{A}=\mathbb{A}_0\leq\mathbb{A}_1\leq\cdots\leq\mathbb{A}_e=\mathbb{E}$ *is a tower of ring extensions where for all* $1\leq i\leq e$ *one of the following holds:*

- $\mathbb{A}_i=\mathbb{A}_{i-1}[t_i]$ *is a ring extension subject to the relation* $t_i^\nu=1$ *for some* $\nu>1$ *where* $\frac{\sigma(t_i)}{t_i}\in\mathbb{A}_{i-1}$ *is a primitive* ν*th root of unity (*t_i *is called an* A-monomial*, and* ν *is called the* order of the A-monomial*);*
- $\mathbb{A}_i=\mathbb{A}_{i-1}[t_i,t_i^{-1}]$ *is a Laurent polynomial ring extension with* $\frac{\sigma(t_i)}{t_i}\in\mathbb{A}_{i-1}^*$*being a unit (*$t_i$ *is called a* P-monomial*);*
- $\mathbb{A}_i=\mathbb{A}_{i-1}[t_i]$ *is a polynomial ring extension with* $\sigma(t_i)-t_i\in\mathbb{A}_{i-1}$ *(*t_i *is called an* S-monomial*).*

Depending on the occurrences of the APS*-monomials such an extension is also called an* A*-/*P*-/*S*-/*AP*-/*AS*/-/*PS-extension*.*

Example 3.6. Take the rational difference ring $(\mathbb{Q}(x),\sigma)$ with $\sigma(x)=x+1$ and $\sigma|_{\mathbb{Q}}=\text{id}$. Then the difference ring $(\mathbb{Q}(x)[s],\sigma)$ with $\sigma(s)=s+\frac{1}{x+1}$ defined in Example 3.1 is an S-extension of $(\mathbb{Q}(x),\sigma)$ and s is an S-monomial over $(\mathbb{Q}(x),\sigma)$.

For the APS-extension $(\mathbb{E},\sigma)$ of $(\mathbb{A},\sigma)$ we will also write $\mathbb{E}=\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle$. Depending on whether t_i with $1\leq i\leq e$ is an A-monomial, a P-monomial or an S-monomial, $\mathbb{G}\langle t_i\rangle$ with $\mathbb{G}=\mathbb{A}\langle t_1\rangle\dots\langle t_{i-1}\rangle$ stands for the algebraic ring extension $\mathbb{G}[t_i]$ with t_i^ν for some $\nu>1$, for the ring of Laurent polynomials $\mathbb{G}[t_1,t_1^{-1}]$ or for the polynomial ring $\mathbb{G}[t_i]$, respectively.

For such a tower of APS-extensions we can use the following lemma iteratively to construct an evaluation function.

Lemma 3.7. *Let* $(\mathbb{A},\sigma)$ *be a difference ring with a subfield* $\mathbb{K}\subseteq\mathbb{A}$ *where* $\sigma|_{\mathbb{K}}=\text{id}$ *that is equipped with an evaluation function* $\text{ev}:\mathbb{A}\times\mathbb{N}\to\mathbb{K}$ *and o-function* L*. Let* $(\mathbb{A}\langle t\rangle,\sigma)$ *be an* APS*-extension of* $(\mathbb{A},\sigma)$ *with* $\sigma(t)=\alpha\,t+\beta$ *(*$\alpha=1$, $\beta\in\mathbb{A}$ *or* $\alpha\in\mathbb{A}^*$, $\beta=0$*). Further, suppose that* $\text{ev}(\sigma^{-1}(\alpha),n)\neq0$ *for all* $n\geq\mu$ *for some* $\mu\in\mathbb{N}$*. Then the following holds.*

1. *Take* $l\in\mathbb{N}$ *with* $l\geq\max(L(\sigma^{-1}(\alpha),L(\sigma^{-1}(\beta)),\mu)$*; if* $t^\lambda=1$ *for some* $\lambda>1$ *(*t *is an* A*-monomial), set* $l=1$*. Then* $\text{ev}':\mathbb{A}\langle t\rangle\times\mathbb{N}\to\mathbb{K}$ *given by*

$$\text{ev}'(\sum_{i=a}^{b}f_i\,t^i,n)=\sum_{i=a}^{b}\text{ev}(f_i,n)\,\text{ev}'(t,n)^i\quad\forall n\in\mathbb{N}\tag{18}$$

with $f_i\in\mathbb{A}$ *for* $a\leq i\leq b$ *and*

$$\text{ev}'(t,n)=\begin{cases}\prod_{i=l}^{n}\text{ev}(\sigma^{-1}(\alpha),i) & \text{if}^{1}\ \sigma(t)=\alpha\,t\\ \sum_{i=l}^{n}\text{ev}(\sigma^{-1}(\beta),i) & \text{if}\ \sigma(t)=t+\beta\end{cases}\tag{19}$$

is an evaluation function for $(\mathbb{A}\langle t\rangle, \sigma)$.

2. *There is an o-function* $L' \colon \mathbb{A}\langle t\rangle \to \mathbb{N}$ *for* ev' *defined by*

$$L'(f) = \begin{cases} L(f) & \text{if } f \in \mathbb{A}, \\ \max(l-1, L(f_a), \dots, L(f_b)) & \text{if } f = \sum_{i=a}^{b} f_i t^i \notin \mathbb{A}\langle t\rangle \setminus \mathbb{A}. \end{cases} \tag{20}$$

Example 3.8. In Example 3.1 we followed precisely the construction (1) of the above lemma to construct for $(\mathbb{Q}(x)[s], \sigma)$ an evaluation function. For this ev we can now apply also the construction (2) to enhance the o function $L : \mathbb{Q}(x) \to \mathbb{N}$ (given in Example 3.3 with $v = 0$) to $L : \mathbb{Q}(x)[h] \to \mathbb{N}$ by setting $L(f) = \max(L(f_0), \dots, L(f_b))$ for $f = \sum_{i=0}^{b} f_i s^i$.

Definition 3.9. *An APS-extension* $(\mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle, \sigma)$ *of* $(\mathbb{G}, \sigma)$ *with* $\mathbb{G} = \mathbb{K}(x)$ *is called hypergeometric APS-extension if*

1. *for any A-monomial* t_i *we have* $\frac{\sigma(t_i)}{t_i} \in \mathcal{R}$;

2. *for any P-monomial* t_i *we have* $\frac{\sigma(t_i)}{t_i} \in \mathbb{K}(x)^*$.

Given the above constructions, we are now ready to state that the representations in $\mathrm{SumProd}(\mathbb{G})$ and in hypergeometric APS-extension are closely related.

• **Observation 1:** Given $\{T_1, \dots, T_e\} \subseteq \Sigma\Pi(\mathbb{G})$, one can construct a hypergeometric APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with ev and L following Lemma 3.7 such that there are $a_1, \dots, a_e \in \mathbb{E}$ and $\delta_1, \dots, \delta_e$ with $\mathrm{ev}(a_i, n) = T_i(n)$.

• **Observation 2:**

$$\begin{tikzcd} \boxed{\begin{array}{l} (\mathbb{E}, \sigma) \text{ with } \mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle \text{ a hypergeometric } APS\text{-extension of } (\mathbb{G}, \sigma) \\ \mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K},\ L : \mathbb{E} \to \mathbb{N} \text{ using Lemma 3.7} \end{array}} \arrow[d, shift right=8ex, "\begin{array}{l} \forall n \geq L(t_i): \\ \mathrm{ev}(t_i, n) = T_i(n) \in \Sigma\Pi(\mathbb{G}) \end{array}"] \\ \boxed{\begin{array}{l} W = \{T_1, \dots, T_e\} \subseteq \Sigma\Pi(\mathbb{G}) \text{ is sum-product reduced and} \\ \text{shift stable: sums/products in } T_i \text{ are from } \{T_1, \dots, T_{i-1}\}. \end{array}} \arrow[u, shift right=8ex] \end{tikzcd}$$

In particular, if $f \in \mathbb{E} \setminus \{0\}$, then we can take $0 \neq F \in \mathrm{SumProd}(\{T_1, \dots, T_e\}, \mathbb{G})$ with $F(n) = \mathrm{ev}(f, n)$ for all $n \geq L(f)$.

Definition 3.10. F *is called the* canonical induced sum-product representation of $f \in \mathbb{E}$ denoted by $\mathrm{expr}(f) = F$.

Example 3.11 (Cont. of Ex. 3.1)**.** For $f = x + \frac{x+1}{x} s^4 \in \mathbb{Q}(x)[s]$ with our evaluation function ev we obtain the canonical induced sum-product expression

$$\mathrm{expr}(f) = F = x \oplus \left(\tfrac{x+1}{x} \odot (\mathrm{Sum}(1, \tfrac{1}{x})^{\textcircled{\scriptsize$\wedge$}} 4\right) \in \mathrm{Sum}(\mathbb{Q}(x)))$$

with $F(n) = \mathrm{ev}(f, n)$ for all $n \geq 1$.

Summary: the naive construction of APS-extensions will not gain any substantial simplification (except a transformation to a sum-product reduced representation). In the next section we will refine this construction further to solve Problem SigmaReduce.

[1]If t is an A-monomial, we have $\mathrm{ev}(t, n) = \alpha^n$.

3.2 The canonical representation in $R\Pi\Sigma$-extensions

Let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{G}\langle t_1\rangle\dots\langle t_e\rangle$ be a hypergeometric APS-extension of $(\mathbb{G},\sigma)$ with $\mathbb{G}=\mathbb{K}(x)$ where ev with L is constructed as given in Lemma 3.7.
Now define $\tau\colon\mathbb{A}\to\mathbb{K}^{\mathbb{N}}$ with

$$\tau(f)=(\mathrm{ev}(f,n))_{n\geq 0}=(\mathrm{ev}(f,0),\mathrm{ev}(f,1),\mathrm{ev}(f,2),\dots). \tag{21}$$

Due to (15) and (16) the map τ can be turned to a ring homomorphism by defining the equivalence relation $(f_n)_{n\geq 0}\equiv(g_n)_{n\geq 0}$ with $f_j=g_j$ for all $j\geq\lambda$ for some $\lambda\in\mathbb{N}$. It is easily seen that the set of equivalence classes $[f]$ with $f\in\mathbb{K}^{\mathbb{N}}$ forms with $[f]+[g]:=[f+g]$ and $[f][g]:=[fg]$ again a commutative ring with the identity element $[\mathbf{1}]$ which we will denote by $S(\mathbb{K})$. In the following we will simply write f instead of $[f]$. In this setting, $\tau\colon\mathbb{A}\to S(\mathbb{K})$ forms a ring homomorphism. In addition the shift operator $S:S(\mathbb{K})\to S(\mathbb{K})$ defined by

$$S((a_0,a_1,a_2,\dots))=(a_1,a_2,a_3,\dots)$$

turns to a ring automorphism. In the following we call $(S(\mathbb{K}),S)$ *also the (difference) ring of sequences over* $\mathbb{K}$. Finally, we observe that property (17) implies that

$$\tau(\sigma(f))=S(\tau(f)) \tag{22}$$

holds for all $f\in\mathbb{A}$, i.e., τ turns to a difference ring homomorphism. Finally, property (14) implies

$$\tau(c)=\boldsymbol{c}=(c,c,c,\dots) \tag{23}$$

for all $c\in\mathbb{K}$. In the following we call a ring homomorphism $\tau:\mathbb{A}\to S(\mathbb{K})$ with (22) and (23) also a $\mathbb{K}$-*homomorphism.*

Lemma 3.12. *Let $W=\{T_1,\dots,T_e\}\in\Sigma\Pi(\mathbb{G})$ with $T_i=expr(t_i)$. Then W is canonical reduced iff τ is injective.*

Proof. Suppose that W is canonical reduced and take $a,b\in\mathbb{E}$ with $\tau(a)=\tau(b)$. Define $A:=\mathrm{expr}(a)$ and $B:=\mathrm{expr}(b)$ with $\mathrm{SumProd}(W,\mathbb{G})$ where $(A(n))_{\geq 0}=\tau(a)$ and $(B(n))_{\geq 0}=\tau(b)$ Then $A(n)=B(n)$ for all $n\geq\delta$ for some $\delta\in\mathbb{N}$. Thus A and B are the same up to permutation of the operands in $\oplus$ and $\odot$. In particular, $a=b$. Thus τ is injective.
Conversely, suppose that τ is injective and take $A,B\in\mathrm{SumProd}(W,\mathbb{G})$ with $A(n)=B(n)$ for all $n\geq\delta$ for some $\delta\in\mathbb{N}$. By construction we can take $a,b\in\mathbb{E}$ with $\mathrm{expr}(a)=A$ and $\mathrm{expr}(b)=B$ with $\tau(a)=(A(n))_{n\geq 0}$ and $\tau(b)=(B(n))_{n\geq 0}$. Thus $\tau(a)=\tau(b)$ and since τ is injective it follows that $a=b$. But this implies that $A=B$. ☐

In this context, the set of constants plays a decisive role.

Definition 3.13. *For a difference ring $(\mathbb{A},\sigma)$ the* set of constants *is defined by* $\mathrm{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}\mid\sigma(c)=c\}$.

In general, $\mathrm{const}_\sigma\mathbb{A}$ is a subring of $\mathbb{A}$. If $\mathbb{A}$ is a field, then $\mathrm{const}_\sigma\mathbb{A}$ itself is a field which one also calls the *constant field of* $(\mathbb{A},\sigma)$.

Lemma 3.14. *For our difference field $\mathbb{G} = \mathbb{K}(x)$ with $\sigma(x) = x + 1$ and $\mathrm{const}_\sigma\mathbb{K} = \mathbb{K}$ we have $\mathrm{const}_\sigma\mathbb{K}(x) = \mathbb{K}$.*

With this extra notion we can state now the following remarkable property.

Theorem 3.15. *Let $(\mathbb{E}, \sigma)$ be a hypergeometric APS-extension of $(\mathbb{G}, \sigma)$ with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ and let τ be the $\mathbb{K}$-homomorphism given by $\tau(f) = (\mathrm{ev}(f, n))_{n\geq 0}$. Then τ is injective iff $\mathrm{const}_\sigma\mathbb{E} = \mathbb{K}$.*

This result gives rise to the following refined definition of PS-field/APS-extensions.

Definition 3.16. *Let $(\mathbb{E}, \sigma)$ be an APS-extension of $(\mathbb{A}, \sigma)$ as defined in Definition 3.5. Then this is called an* $R\Pi\Sigma$-extension *if* $\mathrm{const}_\sigma\mathbb{E} = \mathrm{const}_\sigma\mathbb{A}$.

Example 3.17 (Cont. of Ex. 3.6)**.** Consider the difference ring $(\mathbb{Q}(x)[s], \sigma)$ from Example 3.6. Since $\mathrm{ev} : \mathbb{Q}(x)[s] \to \mathbb{Q}$ defined by (12) and (13) (with $\mathrm{ev}' = \mathrm{ev}$) is an evaluation function of $(\mathbb{Q}(x)[s], \sigma)$ we can construct the $\mathbb{Q}$-homomorphism $\tau : \mathbb{Q}(x)[s] \to \mathbf{S}(\mathbb{Q})$ defined by (21). Since s is a Σ-monomial over $\mathbb{Q}(x)$, we get $\mathrm{const}_\sigma\mathbb{Q}(x)[s] = \mathbb{Q}$. Thus we can apply Theorem 3.15 and it follows that

$$\tau(\mathbb{Q}(x))[\tau(s)] = \tau(\mathbb{Q}(x))[(\mathrm{ev}(s, n))_{n\geq 0}] = \tau(\mathbb{Q}(x))[(S(n))_{n\geq 0}]$$

with $S = \mathrm{expr}(s) = \mathrm{Sum}(1, \frac{1}{x}) \in \Sigma(\mathbb{Q}(x))$ is isomorphic to the polynomial ring $\mathbb{Q}(x)[s]$. Further, $(S(n))_{n\geq 0}$ with $S(n) = \sum_{k=1}^{n} \frac{1}{k}$ is transcendental over $\tau(\mathbb{Q}(x))$.

Remark 3.18. Example 3.17 generalizes as follows. Suppose that we are given a hypergeoemtric $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with

$$\mathbb{G}[\rho_1] \dots [\rho_l][p_1, p_1^{-1}] \dots [p_u, p_u^{-1}][s_1] \dots [s_r]$$

where the ρ_i are R-monomials with $\zeta_i = \frac{\sigma(\rho_i)}{\rho_i} \in \mathcal{R}$ being primitive roots of unity, p_i are Π-monomials and the s_i are Σ-monomials. In addition, take an evaluation function ev with o-function L by iterative applications of Lemmas 3.7. Here we may assume that

- $\mathrm{ev}(\rho_i, n) = \zeta_i^n$ for all $1 \leq i \leq l$,
- $\mathrm{ev}(p_i, n) = P_i(n)$ with $\mathrm{expr}(p_i) = P_i \in \Pi(\mathbb{G})$ for all $1 \leq i \leq u$, and
- $\mathrm{ev}(s_i, n) = S_i(n)$ with $\mathrm{expr}(s_i) = S_i \in \Sigma(\mathbb{G})$ for all $1 \leq i \leq r$.

Then $\tau : \mathbb{E} \to S(\mathbb{K})$ with (21) is a $\mathbb{K}$-homomorphism. By Theorem 3.15 it follows that τ is injective and thus

$$\begin{aligned}
\tau(\mathbb{E}) = \ & \tau(\mathbb{G}) \ [\tau(\rho_1)] \dots [\tau(\rho_l)] \\
& \qquad \times [\tau(p_1), \tau(p_1)^{-1}] \dots [\tau(p_u), \tau(p_u)^{-1}] \\
& \qquad \times [\tau(s_1)] \dots [\tau(s_r)] \\
= \ & \tau(\mathbb{G}) \ [(\zeta_1^n)_{n\geq 0}] \dots [(\zeta_l^n)_{n\geq 0}] \\
& \qquad \times [(P_1(n))_{n\geq 0}, (\tfrac{1}{P_1(n)})_{n\geq 0}] \dots [(P_u(n))_{n\geq 0}, (\tfrac{1}{P_u(n)})_{n\geq 0}] \\
& \qquad \times [(S_1(n))_{n\geq 0}] \dots [(S_r(n))_{n\geq 0}]
\end{aligned}$$

forms a (Laurent) polynomial ring extension over the ring of sequences

$$R = \tau(\mathbb{G})[(\zeta_1^n)_{n\geq 0}]\dots[(\zeta_l^n)_{n\geq 0}].$$

In particular, we conclude that the sequences

$$(P_1(n))_{n\geq 0}, (\tfrac{1}{P_1(n)})_{n\geq 0}, \dots (P_u(n))_{n\geq 0}, (\tfrac{1}{P_u(n)})_{n\geq 0}, (S_1(n))_{n\geq 0}, \dots, (S_r(n))_{n\geq 0}$$

are, up to the trivial relations $(P_i(n))_{n\geq 0}\cdot(\frac{1}{P_i(n)})_{n\geq 0} = 1$ for $1 \leq i \leq u$, algebraically independent among each other over the ring R.

We are now ready to state the main result of this section that connects SumProd($\mathbb{G}$) with difference ring theory.

Theorem 3.19. *Let $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ be in sum-product reduced representation and shift-stable. More precisely, for each $1 \leq i \leq e$ the arising sums/products in T_i are contained in $\{T_1, \dots, T_{i-1}\}$. Then the following two statements are equivalent:*

1. *There is a hypergeometric $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ equipped with an evaluation function* ev *using Lemma 3.7 with $T_i = expr(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.*
2. *W is σ-reduced over $\mathbb{G}$.*

Proof. (1) $\Rightarrow$ (2): By Theorem 3.15 $\tau : \mathbb{E} \to S(\mathbb{K})$ with $\tau(f) = (\text{ev}(f, n))_{n\geq 0}$ is injective und thus by Lemma 3.12 W is cononcial reduced. By assumption W is sum-product reduced and shift-stable and thus W is σ-reduced.
(2) $\Rightarrow$ (1): By Observation 2 we get an APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ equipped with an evaluation function ev using Lemma 3.7 with $T_i = \text{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$. Since W is canonical reduced, τ is injective and thus $\text{const}_\sigma\mathbb{E} = \mathbb{K}$ by Theorem 3.15. Consequently $(\mathbb{E}, \sigma)$ is an $R\Pi\Sigma$-extension of $(\mathbb{G}, \sigma)$. □

This yields immediately the following strategy (actually the only strategy for shift-stable sets).

A Strategy to solve Problem SigmaReduce

Given: $A_1, \dots, A_u \in \text{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.
Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in \text{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct an $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ equipped with an evaluation function $\text{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ and o-function L (using Lemma 3.7) in which $A_1, \dots, A_u$ are modeled by $a_1, \dots, a_u \in \mathbb{E}$. More precisely, for $1 \leq i \leq u$ we compute in addition $\delta_i \in \mathbb{N}$ with $\delta_i \geq L(a_i)$ such that
$$A_i(n) = \text{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{24}$$
2. Set $W = \{T_1, \dots, T_e\}$ with $T_i := \text{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.
3. Set $B_i := \text{expr}(a_i) \in \text{SumProd}(W, \mathbb{G})$ for $1 \leq i \leq u$.
4. Return W, $(B_1, \dots, B_u)$ and $(\delta_1, \dots, \delta_u)$.

What remains open is to enrich this general method with the construction required in step (1). This task will be considered in detail in the next section.