Root Separation Bounds

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Problem: Root Separation Bound

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In : $f \in \mathbb{C}[x]$, square-free

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Out : $B \leq R$ where

- $R = \min_{i \neq j} |\alpha_i - \alpha_j|$ (the root separation)
- α_i are the complex roots of f.

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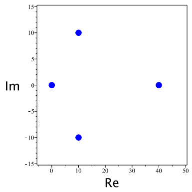
Example

In : $f = x^4 - 60x^3 + 1000x^2 - 8000x$



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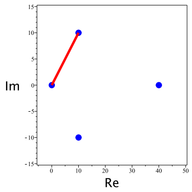

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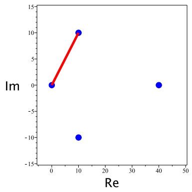
In : $f = x^4 - 60x^3 + 1000x^2 - 8000x$

Out : $B =$ any number $\leq \sqrt{20}$
since $R = \sqrt{20}$



Motivation

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Fundamental problem in algorithmic mathematics.

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Fundamental problem in algorithmic mathematics.

Sign evaluation of algebraic expressions

Real root counting

Root isolation

Algebraic number theory

Topological properties of curves

Quantifier elimination

...

Previous works

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Bounds	1964	Mahler
	1973	Mignotte (also, 1988,1995,2000..)
	1974	Collins (also 2001), Horowitz
	1979	Rump
	2000	Mehlorn, Schirra
	2004	Bughead, Mignotte
	2004	Sasaki
	2006	Schonhage
	2006	Tsigaridas, Emiris
	2008	Muresan
	2008	Batra
	2008	Burnikel, Fleisher, Mehlhorn, Schirra

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	2008	Muresan
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	2008	Burnikel, Fleisher, Mehlhorn, Schirra
Applications	2006	Emeris, Tsigaridas
	2007	Du, Sharma, Yap
	2010	Emiris, Mourrain, Tsigaridas
	2011	Strzebonksi, Tsigaridas
	2012	Tsigaridas

Challenge

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$$B_{Mahler} = \left(\frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \; ||f||_2} \right)^{d-1}$$

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Let $f(x) = x^4 - 60x^3 + 1000x^2 - 8000x$

Then

$$B_{Mahler}(f(x)) \qquad = \qquad 8.26 \times 10^{-6} \qquad\qquad \sqrt{20}$$

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Find a bound B such that

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Find a bound B such that

$B(f) \gg B_{Mahler}(f)$ "Better than Mahler"

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Find a bound B such that

$$\begin{array}{lcll} B(f) & \gg & B_{Mahler}(f) & \text{“Better than Mahler”} \\ B(f(x-t)) & = & B(f(x)) & \text{“Invariant under translation”} \end{array}$$

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Find a bound B such that

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$B(f(x-t))$	$=$	$B(f(x))$	"Invariant under translation"
$B(f(x/s))$	$=$	$sB(f(x))$	"Covariant under scaling"

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Theorem

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Let

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \;\; H^{d-1}}$$

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- $\tilde{a}_i \quad = \quad$ *i-th coefficient of* $f(x - a_{d-1}/d)$

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- B_{New} *is a root separation bound for f*
- B_{New} *is invariant under translation*

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Then

- B_{New} *is a root separation* *bound* *for* f
- B_{New} *is* *invariant* *under translation*
- B_{New} *is* *covariant* *under scaling*

Comparison: Magnitude

Comparison: Magnitude

Bounds

Mahler **New**

Comparison: Magnitude

Bounds

Mahler

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$H = 2.17 \times 10^2$

Comparison: Magnitude

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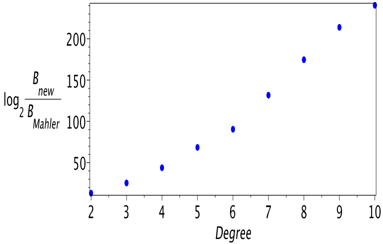
$d = \text{degree}$

$B\text{-height} = max_i \left(|a_i| / \binom{d}{i} \right)^{\frac{1}{d-i}}$

Comparison: Magnitude

d = degree

B-height $= max_i \left(|a_i| / \binom{d}{i} \right)^{\frac{1}{d-i}}$



Fix B-height $= 2^{17}$. Change d.

Comparison: Magnitude

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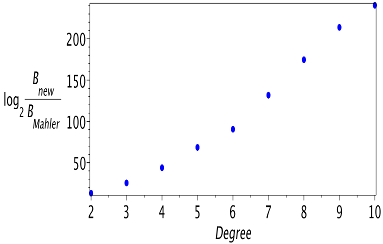

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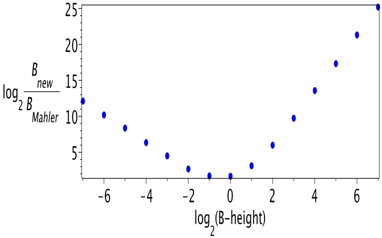

Fix $d = 3$. Change B-height.

Comparison: Magnitude

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Mignotte polynomials have small separation bounds.

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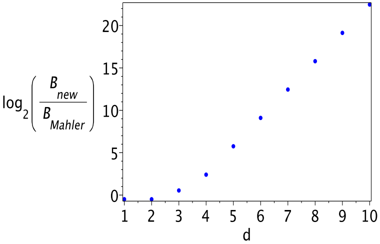

Fix $h = 10$. Change d.

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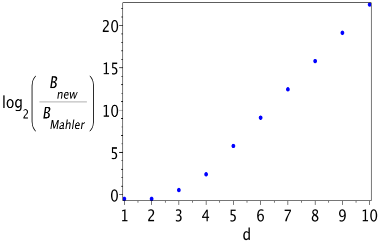

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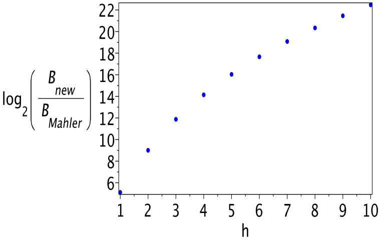

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Fix f. Consider $f(x + t)$ for various t.

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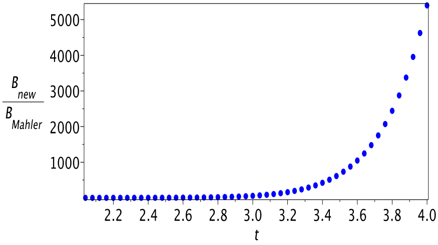

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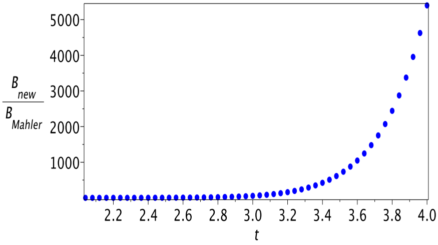

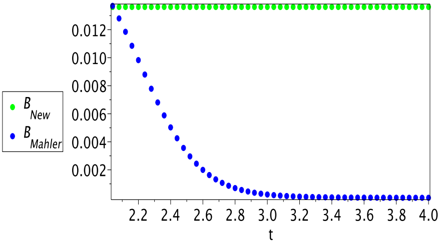

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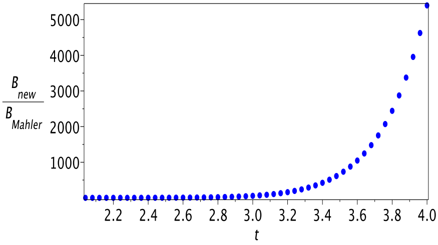

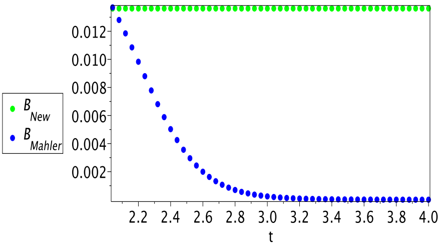

Note

B_{New} is invariant under translation.

Comparison: Behaviour under Scaling

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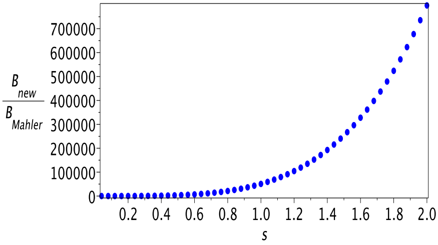

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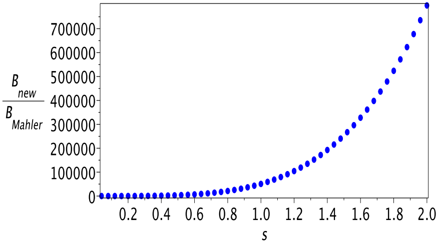

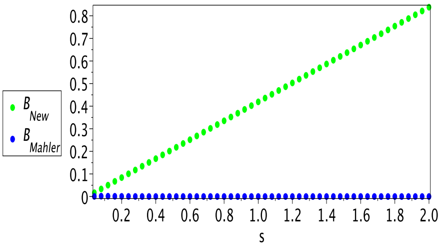

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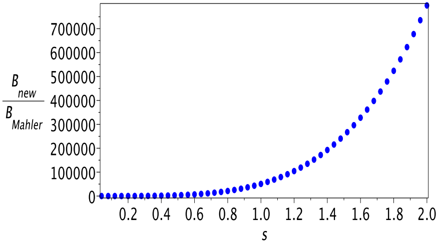

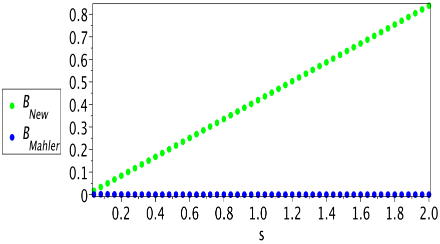

Note

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Derivation of New Bound

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- We choose B to be $B_{Mahler\infty}$.

$$B_{Mahler\infty} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \left(\sqrt{d+1}\ ||f||_\infty\right)^{d-1}}$$

where $||f||_\infty = \max_i ||a_i|$.

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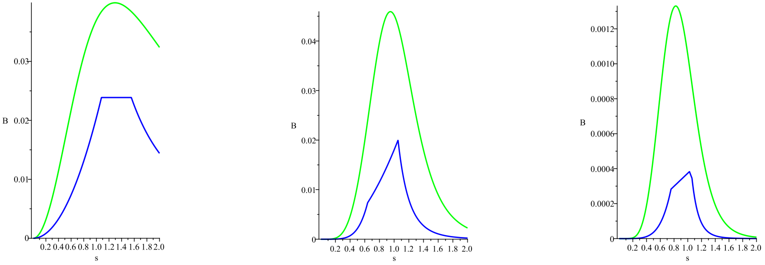

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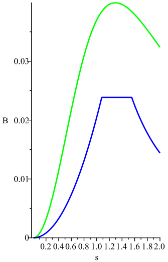

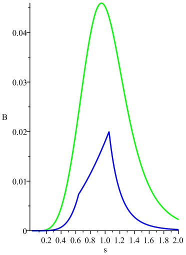

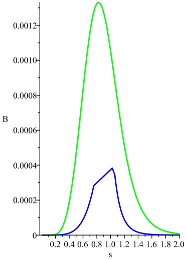

- B_{Mahler} [-] B_{Mahler_∞} [-]

- $B_{Mahler}(s)$ and $B_{Mahler_\infty}(s)$ behave similarly.

Derivation of New Bound

- Maximize $B_{Mahler_{\infty}}(s)$.

Derivation of New Bound

- Maximize $B_{Mahler\infty}(s)$.

$$\max_s B_{Mahler\infty}(s) = \max_s \frac{1}{s} \frac{\sqrt{3|dis(f_s)|}}{d^{d/2+1}\sqrt{d+1}^{d-1}||f_s||_\infty^{d-1}}$$

$$= \frac{\sqrt{3|dis(f)|}}{d^{d/2+1}\sqrt{d+1}^{d-1}\min_s \max_i\{s^{g(i)}|a_i|^{d-1}\}}$$

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- Find

$$Q = \min_{s>0} \max_i s^{g(i)} |a_i|^{d-1}$$

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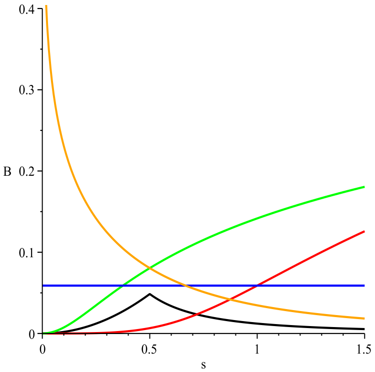

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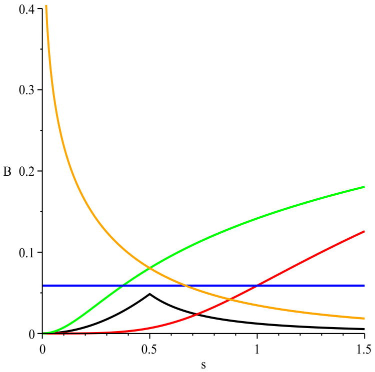

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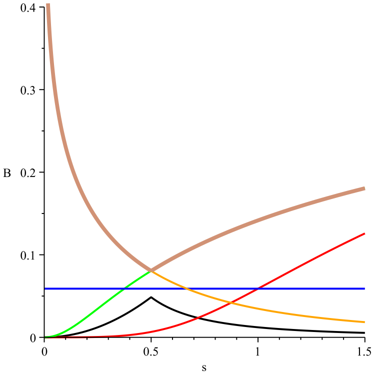

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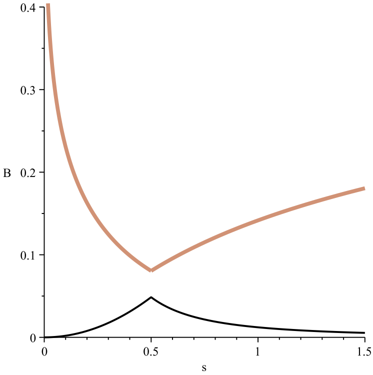

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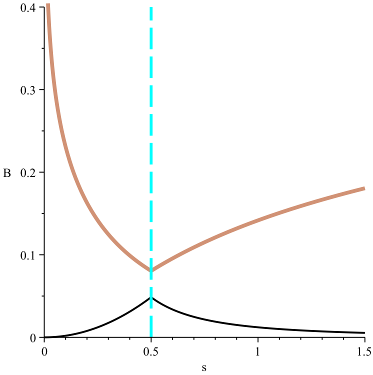

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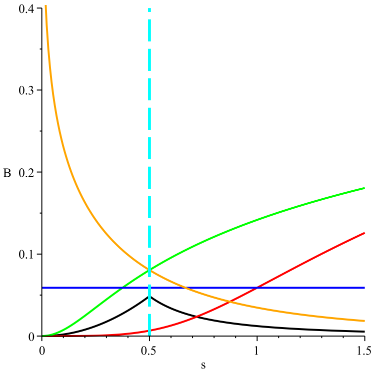

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- Apply this bound to $f(x - \frac{a_{d-1}}{d})$, finally obtaining

$$B_{New} = \frac{\sqrt{3|dis(f)|}}{d^{d/2+1} \quad H^{d-1}}$$

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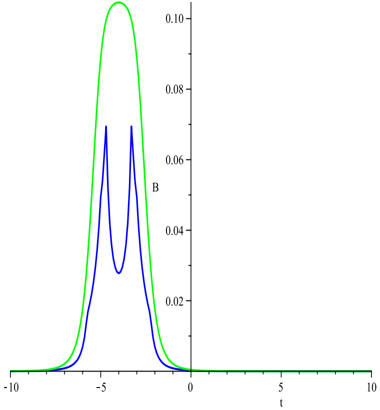

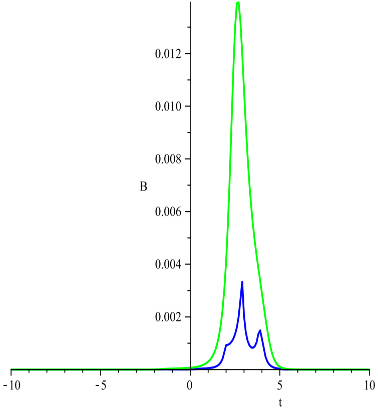

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- Optimize for root translation-scaling. $f(sx+t)$.