Assignment 1 (by April 29, 2019)

Exercise 1. Compute the formulae relating angular velocity, angular momentum, and kinetic energy, for the case where the reference point is not equal to the center of mass. One of these two points (make a choice!) may be assumed to be the origin.

Exercise 2. Which positive semidefinite matrices are moments of inertia?

1. Let (v_1, v_2, v_3) be an orthonormal bases. Let
$$I_{xyz} := \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}.$$
Show that $\langle v_3 \mid I_{xyz} v_3 \rangle \le \langle v_1 \mid I_{xyz} v_1 \rangle + \langle v_2 \mid I_{xyz} v_2 \rangle$.

2. Define
$$\mathcal{I} := \{M \in \mathbb{R}^{3\times 3} \mid M \text{ in symmetric and positive semidefinite and}$$
$$\forall v_1, v_2, v_3 : (v_1, v_2, v_3) \text{ form an ONB } \implies \langle v_3 \mid M v_3 \rangle \le \langle v_1 \mid M v_1 \rangle + \langle v_2 \mid M v_2 \rangle\}.$$
Prove that every moment of intertia is in $\mathcal{I}$.

3. Let $0 \le a \le b \le c$ be the three eigenvalues of a matrix in $\mathcal{I}$ (multiple eigenvalues are repeated). Prove that $a + b \ge c$.

4. Let $I \in \mathbb{R}^{3\times 3}$ be a symmetric positive semidefinite matrix with eigenvalues $0 \le a \le b \le c$ such that $a + b \le c$. Construct a rigid body consisting of six point masses, two in the direction of each eigenvector, such that I is the moment of inertia of that rigid body.

Exercise 3. Prove Lemma 3.1 from the notes on rigid body dynamics:
Let L_1, L_2 be skew lines. Let p be a point which is not in L_1 nor in L_2, such that the line parallel to L_1 through p does not meet L_2 and the line parallel to L_2 through p does not meet L_1. Then there is a unique line through p meeting both L_1 and L_2.

Exercise 4. Is the following 4R-loop balanced with base link 1?

1. In link 1, the axes of the joint 4 is $(x, y, z) = (t, 0, 0)_t$, with assembly points $A_{14} = (0, 0, 0)$ and $B_{14} = (5, 0, 0)$; and the axes of joint 1 is $(x, y, z) = (3t, 10, 4t)_t$, with assembly points $A_{12} = (0, 10, 0)$ and $B_{12} = (3, 10, 4)$. The mass of link 1 is 3, and the center of mass is $(4, 3, 2)$.

2. In link 2, the axes of the joint 1 is $(x, y, z) = (t, 0, 0)_t$, with assembly points $A_{21} = (0, 0, 0)$ and $B_{21} = (5, 0, 0)$; and the axes of joint 2 is $(x, y, z) = (3t, 10, -4t)_t$ with assembly points $A_{23} = (0, 10, 0)$ and $B_{23} = (3, 10, -4)$. The mass of link 2 is 1, and the center of mass is $(0, -10, 12)$.

3. In link 3, the axes of the joint 2 is $(x, y, z) = (t, 0, 0)_t$, with assembly points $A_{32} = (0, 0, 0)$ and $B_{32} = (5, 0, 0)$; and the axes of joint 3 is $(x, y, z) = (3t, 10, 4t)_t$, with assembly points $A_{34} = (0, 10, 0)$ and $B_{34} = (3, 10, 4)$. The mass of link 3 is 2, and the center of mass is $(9, 5, 2)$.

4. In link 4, the axes of the joint 3 is $(x, y, z) = (t, 0, 0)_t$, with assembly points $A_{43} = (0, 0, 0)$ and $B_{43} = (5, 0, 0)$; and the axes of joint 4 is $(x, y, z) = (3t, 10, -4t)_t$, with assembly points $A_{41} = (0, 10, 0)$ and $B_{41} = (3, 10, -4)$. The mass of link 4 is 1, and the center of mass is $(6, 20, -8)$.

The mechanism is assembled so that $A_{14} = A_{41}, B_{14} = B_{41}$ etc. It can be shown that the linkage is actually moving.
After redistributing the masses to the axes, we obtain two points on each axes with (maybe negative) points: on joint 2, we get point p_{23} with mass m_{23} and point p_{32} with mass m_{32}, and on joint 3 we get point p_{34} with mass m_{34} and point p_{43} with mass m_{43}. We also get two points on each of the other two joints, but they are not moving when link 1 is the base link, so we ignore them. Note that $m_{32} + m_{34} > 0$ because this sum is the mass of link 3. The 4R loop is perfectly balanced if $p_{23} = p_{32}$ and $p_{34} = p_{43}$ and $m_{23} + m_{32} = 0$ and $m_{34} + m_{43} = 0$.

Exercise (computation of the decoding matrix in superdence coding): Assume that Alice and Bob share a pair of qubits XY in Bell state $b := \frac{\boxed{0|1} - \boxed{1|0}}{\sqrt{2}}$, where Alice controls X and Bob controls Y. Alice transmits two bits to Bob by applying a unitary transformation depending on the two bits, and sending X to Bob. Here is how it works:

00 : Alice applies $C_{00} := I_2$ (i.e., she leaves X as it is).

01 : Alice applies $C_{01} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

10 : Alice applies $C_{10} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

11 : Alice applies $C_{11} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Bob receives X from Alice and applies a certain unitary matrix D to the pair XY and then measures X and Y. For the correct choice of D, the result is equal to the transmitted code word. Compute $c_{ij} := (C_{ij} \otimes I_2)b$ for $i, j = 0, 1$. Compute a matrix D such that

$$Dc_{00} = \boxed{0|0}, Dc_{01} = \boxed{0|1}, Dc_{10} = \boxed{1|0}, Dc_{11} = \boxed{1|1}.$$

Show that D is unitary.