FORMAL MODELLING

Modelling Problems in Geometry and Discrete Mathematics

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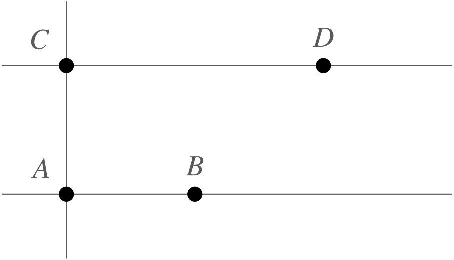
AN INTRODUCTORY EXAMPLE

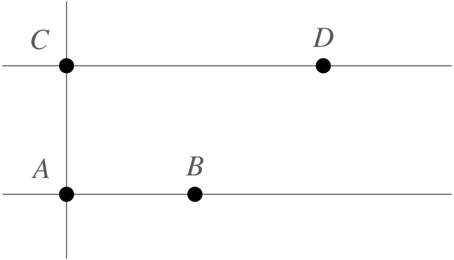
- Given two points A and C and the line passing through A and C.
- Given a point B such that the line *AB* is perpendicular to the line *AC*.
- Given a point D such that the line *CD* is perpendicular to the line *AC*.



Then

- the lines *AB* and *CD* must be parallel.

AN INTRODUCTORY EXAMPLE



The logical statement describing the geometric situation is

$$\forall_{A,B,C,D} \left((\text{perpendicular}(A,B,A,C) \wedge \text{perpendicular}(C,A,C,D)) \Rightarrow \text{parallel}(A,B,C,D) \right)$$

with appropriate predicates ‘perpendicular’ and ‘parallel’.

MODELLING GEOMETRY IN ALGEBRA

1. Introduce a coordinate system:

$$A = (0,0) \qquad B = (b_1, b_2) \qquad C = (c_1, c_2) \qquad D = (d_1, d_2).$$

2. Define predicates:

$$\text{perpendicular}(A, B, A, C): \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cdot \begin{pmatrix} c1 \\ c_2 \end{pmatrix} = \underbrace{b_1 c_1 + b_2 c_2}_{p_1} = 0$$

$$\text{perpendicular}(C, A, C, D): \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cdot \begin{pmatrix} d_1 - c_1 \\ d_2 - c_2 \end{pmatrix} = \underbrace{c_1(d_1 - c_1) + c_2(d_2 - c_2)}_{p_2} = 0$$

$$\text{parallel}(A, B, C, D): \quad \begin{pmatrix} b_2 \\ -b_1 \end{pmatrix} \cdot \begin{pmatrix} d_1 - c_1 \\ d_2 - c_2 \end{pmatrix} = \underbrace{b_2(d_1 - c_1) - b_1(d_2 - c_2)}_{p_3} = 0$$

TRANSFORM PROOF GOAL

$$\forall_{b_1,b_2,c_1,c_2,d_1,d_2} (p_1 = 0 \land p_2 = 0 \Rightarrow p_3 = 0)$$

$\downarrow$ de'Morgan's rule

$$\neg \exists_{b_1,b_2,c_1,c_2,d_1,d_2} p_1 = 0 \land p_2 = 0 \land p_3 \neq 0$$

$\downarrow$ trick!

$$\neg \exists_{b_1,b_2,c_1,c_2,d_1,d_2} p_1 = 0 \land p_2 = 0 \land \exists_{\alpha_0} \alpha_0 p_3 - 1 = 0$$

$\downarrow \alpha_0 \neq$ all other variables

$$\neg \exists_{b_1,b_2,c_1,c_2,d_1,d_2,\alpha_0} p_1 = 0 \land p_2 = 0 \land \alpha_0 p_3 - 1 = 0$$

PROOF VS. SYSTEM OF EQUATIONS

$$\neg \exists_{b_1,b_2,c_1,c_2,d_1,d_2,\alpha_0} \; p_1 = 0 \wedge p_2 = 0 \wedge \alpha_0 p_3 - 1 = 0$$

just expresses that the system of polynomial (algebraic) equations

$$\begin{aligned} p_1 &= 0 \\ p_2 &= 0 \\ \alpha_0 p_3 - 1 &= 0 \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} b_1 c_1 + b_2 c_2 &= 0 \\ c_1 d_1 - c_1^2 + c_2 d_2 - c_2^2 &= 0 \\ \alpha_0 b_2 d_1 - \alpha_0 b_2 c_1 - \alpha_0 b_1 d_2 + \alpha_0 b_1 c_2 - 1 &= 0 \end{aligned}$$

has no solutions!

SOLVING SYSTEM OF EQUATIONS

Using Mathematica, we get a solution

$$c_1 = 0 \qquad c_2 = 0 \qquad b_1 = 0 \qquad b_2 = 1 \qquad d_1 = 1 \qquad d_2 = 0 \qquad \alpha_0 = 1.$$

This means, we have no proof of our statement!

In fact, the solution

$$A = C \qquad B = (0, 1) \qquad D = (1, 0)$$

gives a counterexample.

The ‘line passing through A and C’ $\rightsquigarrow$ a point.

‘perpendicular(A, B, A, C)’ and ‘perpendicular(C, A, C, D)’ trivially become true.

‘parallel(A, B, C, D)’ is false because *AB* and *CD* are perpendicular.

IMPROVED APPROACH

We used an inaccurate model: 'the line passing through A and C' $\rightsquigarrow A \neq C$.

$A \neq C$ means $c_1 \neq 0 \vee c_2 \neq 0$, i.e.

$$\exists_{\alpha_1} \alpha_1 c_1 - 1 = 0 \vee \exists_{\alpha_2} \alpha_2 c_2 - 1 = 0$$

$$\Updownarrow$$

$$\exists_{\alpha_1, \alpha_2} (\alpha_1 c_1 - 1)(\alpha_2 c_2 - 1) = 0$$

IMPROVED MODEL

Theorem proved iff

$$\begin{aligned} b_1c_1 + b_2c_2 &= 0 \\ c_1d_1 - c_1^2 + c_2d_2 - c_2^2 &= 0 \\ (\alpha_1c_1 - 1)(\alpha_2c_2 - 1) &= 0 \\ \alpha_0b_2d_1 - \alpha_0b_2c_1 - \alpha_0b_1d_2 + \alpha_0b_1c_2 - 1 &= 0 \end{aligned}$$

has no solution.

Mathematica confirms that there is no solution $\rightsquigarrow$ original statement proved.

THE GENERAL MODEL

Let us assume we have a geometrical configuration described by

$$p_1 = 0 \qquad \dots \qquad p_n = 0 \qquad q_1 \neq 0 \qquad \dots \qquad q_m \neq 0$$

and a conclusion described by $c = 0$, where $p_i, q_j, c \in \mathbb{Q}[x_1, \dots, x_l]$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Then

$$\forall_{x_1,\dots,x_l} (p_1 = 0 \land \dots \land p_n = 0 \land q_1 \neq 0 \land \dots \land q_m \neq 0 \Rightarrow c = 0)$$

$$\Updownarrow$$

$$\neg \exists_{x_1,\dots,x_l} \neg(p_1 = 0 \land \dots \land p_n = 0 \land q_1 \neq 0 \land \dots \land q_m \neq 0 \Rightarrow c = 0)$$

$$\Updownarrow$$

$$\neg \exists_{x_1,\dots,x_l} p_1 = 0 \land \dots \land p_n = 0 \land q_1 \neq 0 \land \dots \land q_m \neq 0 \land c \neq 0$$

RABINOVICH-TRICK

The negated equalities can then be turned into equalities:

$$\neg \underset{x_1,\ldots,x_l}{\exists} p_1 = 0 \wedge \ldots \wedge p_n = 0 \wedge \underset{\alpha_1}{\exists} \alpha_1 q_1 - 1 = 0 \wedge \ldots \wedge \underset{\alpha_m}{\exists} \alpha_m q_m - 1 = 0 \wedge \underset{\alpha_0}{\exists} \alpha_0 c - 1 = 0$$

$\Updownarrow$ $\alpha_0, \alpha_1, \ldots, \alpha_m$ are new variables

$$\neg \underset{x_1,\ldots,x_l,\alpha_0,\alpha_1,\ldots,\alpha_m}{\exists} p_1 = 0 \wedge \ldots \wedge p_n = 0 \wedge \alpha_1 q_1 - 1 = 0 \wedge \ldots \wedge \alpha_m q_m - 1 = 0 \wedge \alpha_0 c - 1 = 0$$

PROOF VS. SYSTEM OF EQUATIONS

$$\neg \underset{x_1,\ldots,x_l,\alpha_0,\alpha_1,\ldots,\alpha_m}{\exists} p_1 = 0 \wedge \ldots \wedge p_n = 0 \wedge \alpha_1 q_1 - 1 = 0 \wedge \ldots \wedge \alpha_m q_m - 1 = 0 \wedge \alpha_0 c - 1 = 0$$

means that ...

the system of polynomial equations in the variables $x_1, \ldots, x_l, \alpha_0, \alpha_1, \ldots, \alpha_m$

$$p_1 = 0 \quad \ldots \quad p_n = 0$$
$$\alpha_1 q_1 - 1 = 0 \quad \ldots \quad \alpha_m q_m - 1 = 0$$
$$\alpha_0 c - 1 = 0$$

has no solutions for $x_1, \ldots, x_l, \alpha_0, \alpha_1, \ldots, \alpha_m$.

EXAMPLE: THEOREM OF THALES

Let A and B be two points and M the midpoint between A and B. Let c be the circle with center M through A and B, and let C be any point on c. Then *AC* and *BC* are perpendicular.

Coordinates: $M = (0, 0), A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2)$.

$$a_1 + b_1 = 0 \qquad a_2 + b_2 = 0 \qquad \text{(M is the midpoint between A and B)}$$

$$a_1^2 + a_2^2 - c_1^2 - c_2^2 = 0 \qquad \text{(distance from A and C to M must be equal)}$$

$$(c_1 - a_1)(c_1 - b_1) + (c_2 - a_2)(c_2 - b_2) = 0 \qquad \text{(AC and BC are perpendicular)}$$

EXAMPLE: THEOREM OF THALES – EQUATIONS

$$
\begin{aligned}
a_1^2 + a_2^2 - c_1^2 - c_2^2 &= 0 \\
a_1 + b_1 &= 0 \\
a_2 + b_2 &= 0 \\
\alpha_0((c_1 - a_1)(c_1 - b_1) + (c_2 - a_2)(c_2 - b_2)) - 1 &= 0
\end{aligned}
$$

No solution, because equations 2 and 3 mean $a_1 = -b_1$ and $a_2 = -b_2$.

Substitution in equation 4:

$$\alpha_0(\underbrace{c_1^2 - a_1^2 + c_2^2 - a_2^2}_{=0 \text{ by equation 1}}) - 1 = 0, \quad \text{i.e.} \quad -1 = 0 \leadsto \text{ no solution}$$

SOLVABILITY OF SYSTEMS OF POLYNOMIAL EQUATIONS

Given a set of polynomials G, a Gröbner basis of G is a set of polynomials B, s.t.

$$\underset{x_1,\ldots,x_n}{\forall} \left(\underset{g\in G}{\forall} g(x_1,\ldots,x_n) = 0 \Leftrightarrow \underset{b\in B}{\forall} b(x_1,\ldots,x_n) = 0 \right),$$

and B has some special properties that make the system $\underset{b\in B}{\forall} b(x_1,\ldots,x_n) = 0$ 'easier to solve' than the original system $\underset{g\in G}{\forall} g(x_1,\ldots,x_n) = 0$.

Systems of Polynomial Equations	Systems of Linear Equations
Gröbner basis of G	triangular form of a matrix

COMPUTING GRÖBNER BASES

Bruno Buchberger (the founder of RISC) $\overset{1965}{\rightsquigarrow}$ algorithm that computes a Gröbner basis for any given set of polynomials G.

Similar to Gaussian elimination: eliminate variables step-by-step by polynomial reduction.

Polynomial reduction: Subtract multiples of one polynomial from other polynomials in order to cancel terms. Generalization of the univariate polynomial division to multivariate polynomials. Details see “Computer Algebra”.

Implemented in every computer algebra system (like Mathematica, Maple, or Sage).

GRÖBNER BASES AND SYSTEMS OF EQUATIONS

Theorem

A system of polynomial equations

$$g_1 = 0, \quad \ldots \quad , g_n = 0$$

has no solutions over $\mathbb{C}$ if and only if the Gröbner basis of $\{g_1, \ldots, g_n\}$ contains a constant polynomial unequal to 0.

Hence, solvability of systems of polynomial equations can be decided by Gröbner bases.

EXAMPLE: THALES WITH GRÖBNER BASIS

Using Mathematica, we compute

$$\texttt{GroebnerBasis}[\{a_1^2 + a_2^2 - c_1^2 - c_2^2, a_1 + b_1, a_2 + b_2,\\ \alpha_0((c_1 - a_1)(c_1 - b_1) + (c_2 - a_2)(c_2 - b_2)) - 1\}, \{a_1, a_2, b_1, b_2, c_1, c_2, \alpha_0\}]$$

and the answer is $\{1\}$, thus, the Gröbner basis contains the constant polynomial 1 and the system of equations corresponding to the Theorem of Thales is unsolvable, therefore the theorem is proven.

POINTS ON A LINE

$$X_1 = \binom{x_1}{y_1} \qquad X_2 = \binom{x_2}{y_2} \qquad X_3 = \binom{x_3}{y_3} \qquad X_4 = \binom{x_4}{y_4}.$$

Theorem

X_1, X_2, X_3 (and X_4) are collinear if and only if

$$\det\left(\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\right) = 0 \quad \textit{respectively} \quad \det\left(\begin{pmatrix} 1 & x_1 & y_1 & {x_1}^2 + {y_1}^2 \\ 1 & x_2 & y_2 & {x_2}^2 + {y_2}^2 \\ 1 & x_3 & y_3 & {x_3}^2 + {y_3}^2 \\ 1 & x_4 & y_4 & {x_4}^2 + {y_4}^2 \end{pmatrix}\right) = 0.$$

PERPENDICULAR AND PARALLEL

Theorem

1. *X_1X_2 and X_3X_4 are perpendicular if and only if*

$$(x_2 - x_1)(x_4 - x_3) + (y_2 - y_1)(y_4 - y_3) = 0.$$

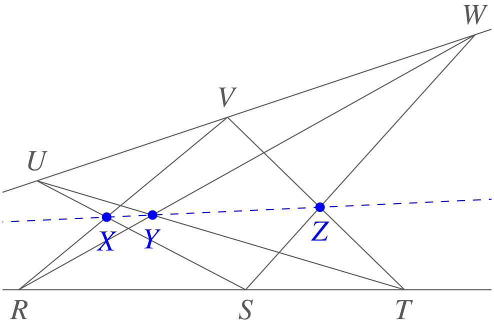
2. *X_1X_2 and X_3X_4 are parallel if and only if*

$$(y_2 - y_1)(x_4 - x_3) - (x_2 - x_1)(y_4 - y_3) = 0.$$

EXAMPLE: THEOREM OF PAPPUS

Given one set of collinear points R, S, and T, and another set of collinear points U, V, and W, s.t. R, S, and U and R, S, and V, respectively, are not collinear.

Then the intersection points X, Y, and Z of line pairs *RV* and *SU*, *RW* and *TU*, *SW* and *TV* are collinear.



POLYNOMIAL MODEL

$$R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \qquad S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \qquad T = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

$X = RV \cap SU$: R, V, and X as well as S, U, and X are collinear, i.e.

$$\det(\begin{pmatrix} 1 & r_1 & r_2 \\ 1 & v_1 & v_2 \\ 1 & x_1 & x_2 \end{pmatrix}) = -r_2v_1 + r_1v_2 + r_2x_1 - r_1x_2 - v_2x_1 + v_1x_2 = 0$$

$$\det(\begin{pmatrix} 1 & s_1 & s_2 \\ 1 & u_1 & u_2 \\ 1 & x_1 & x_2 \end{pmatrix}) = -s_2u_1 + s_1u_2 + s_2x_1 - s_1x_2 + u_1x_2 - u_2x_1 = 0$$

POLYNOMIAL MODEL: $n = 8$, $m = 2$, AND $l = 18$

$$
\begin{aligned}
-r_2v_1 + r_1v_2 + r_2x_1 - r_1x_2 + v_1x_2 - v_2x_1 &= 0\\
-s_2u_1 + s_1u_2 + s_2x_1 - s_1x_2 + u_1x_2 - u_2x_1 &= 0\\
-r_2w_1 + r_1w_2 + r_2y_1 - r_1y_2 + w_1y_2 - w_2y_1 &= 0\\
-t_2u_1 + t_1u_2 + t_2y_1 - t_1y_2 + u_1y_2 - u_2y_1 &= 0\\
-s_2w_1 + s_1w_2 + s_2z_1 - s_1z_2 + w_1z_2 - w_2z_1 &= 0\\
-t_2v_1 + t_1v_2 + t_2z_1 - t_1z_2 + v_1z_2 - v_2z_1 &= 0\\
-r_2s_1 + r_1s_2 + r_2t_1 - r_1t_2 + s_1t_2 - s_2t_1 &= 0\\
-u_2v_1 + u_1v_2 + u_2w_1 - u_1w_2 + v_1w_2 - v_2w_1 &= 0\\
\alpha_1\left(-r_2s_1 + r_1s_2 + r_2u_1 - r_1u_2 + s_1u_2 - s_2u_1\right) - 1 &= 0\\
\alpha_2\left(-r_2s_1 + r_1s_2 + r_2v_1 - r_1v_2 + s_1v_2 - s_2v_1\right) - 1 &= 0\\
\alpha_0\left(-x_2y_1 + x_1y_2 + x_2z_1 - x_1z_2 + y_1z_2 - y_2z_1\right) - 1 &= 0
\end{aligned}
$$

Gröbner basis = $\{1\}$ $\rightsquigarrow$ Theorem of Pappus proved!

ARBITRARY BOOLEAN COMBINATIONS

Same technique can be applied to arbitrary universally quantified boolean combinations of polynomial equalities (negated equalities: $p \neq 0 \equiv \neg(p = 0)$).

$$\text{Prove:} \quad \underset{x_1,\ldots,x_l}{\forall} \Phi,$$

where Φ is an arbitrary boolean combination of polynomial equations with polynomials in $\mathbb{Q}[x_1, \ldots, x_l]$. Rewrite the statement as

$$\neg \underset{x_1,\ldots,x_l}{\exists} \neg\Phi,$$

and then convert $\neg\Phi$ into conjunctive normal form resulting in

$$\neg \underset{x_1,\ldots,x_l}{\exists} (\Phi_{1,1} \vee \ldots \vee \Phi_{1,j_1}) \wedge \ldots \wedge (\Phi_{n,1} \vee \ldots \vee \Phi_{n,j_n}),$$

where each $\Phi_{i,j}$ has the form either $P_{i,j} = 0$ or $\neg(P_{i,j} = 0)$.

FROM CONJUNCTIVE NORMAL FORM TO EQUATIONS

$$Q_{i,j} := \begin{cases} P_{i,j} & \text{if } \Phi_{i,j} \text{ has the form } P_{i,j} = 0 \\ \alpha_{i,j} P_{i,j} - 1 & \text{if } \Phi_{i,j} \text{ has the form } \neg(P_{i,j} = 0) \end{cases}$$

with new variables $\alpha_{i,j}$ (existentially quantified! See Rabinovich-Trick). Since the $\alpha_{i,j}$ are new and distinct from $x_1, \ldots, x_l$:

$$\neg \underset{x_1,\ldots,x_l,\{\alpha_{i,j}\}}{\exists} (Q_{1,1} = 0 \vee \ldots \vee Q_{1,j_1} = 0) \wedge \ldots \wedge (Q_{n,1} = 0 \vee \ldots \vee Q_{n,j_n} = 0).$$

and finally

$$\neg \underset{x_1,\ldots,x_l,\{\alpha_{i,j}\}}{\exists} (Q_{1,1} \cdot \ldots \cdot Q_{1,j_1} = 0) \wedge \ldots \wedge (Q_{n,1} \cdot \ldots \cdot Q_{n,j_n} = 0).$$

THE FINAL SYSTEM OF EQUATIONS

The original statement is equivalent to the unsolvability of the system of polynomial equations

$$\begin{array}{rcl} Q_{1,1} \cdot \ldots \cdot Q_{1,j_1} & = & 0 \\ \vdots & \vdots & \vdots \\ Q_{n,1} \cdot \ldots \cdot Q_{n,j_n} & = & 0, \end{array}$$

which can be decided by computing

$$B = \texttt{GroebnerBasis}[\{Q_{1,1} \cdot \ldots \cdot Q_{1,j_1}, \ldots, Q_{n,1} \cdot \ldots \cdot Q_{n,j_n}\}]$$

and checking, whether B contains a constant polynomial unequal to 0.

EXAMPLE

Generalization of introductory example: if we have two perpendicular lines, then being parallel to one of them is the same as being perpendicular to the other.

With appropriate side-conditions:

$$\forall_{A,B,C,D}\; A \neq C \wedge A \neq B \wedge \text{perpendicular}(A, B, A, C) \Rightarrow \\ \text{perpendicular}(C, A, C, D) \Leftrightarrow \text{parallel}(A, B, C, D)$$

EXAMPLE

$$A = (0,0) \qquad B = (b_1, b_2) \qquad C = (c_1, c_2) \qquad D = (d_1, d_2)$$

Conjunctive normal form of the negated expression inside the quantifier gives

$$(b_1 \neq 0 \lor b_2 \neq 0) \land (c_1 \neq 0 \lor c_2 \neq 0) \land$$
$$\land (\neg \text{parallel}(A, B, C, D) \lor \neg \text{perpendicular}(C, A, C, D)) \land$$
$$\land (\text{parallel}(A, B, C, D) \lor \text{perpendicular}(C, A, C, D)) \land$$
$$\land \text{ perpendicular}(A, B, A, C)$$

Written out:

$$(b_1 \neq 0 \lor b_2 \neq 0) \land (c_1 \neq 0 \lor c_2 \neq 0) \land$$
$$\land (b_2 (d_1 - c_1) - b_1 (d_2 - c_2) \neq 0 \lor -c_1 (d_1 - c_1) - c_2 (d_2 - c_2) \neq 0) \land$$
$$\land (b_2 (d_1 - c_1) - b_1 (d_2 - c_2) = 0 \lor -c_1 (d_1 - c_1) - c_2 (d_2 - c_2) = 0) \land$$
$$\land\, b_1 c_1 + b_2 c_2 = 0.$$

EXAMPLE

Rabinovich-Trick + disjunctions $\longrightarrow$ products

$$
\begin{gathered}
\{-\alpha_0 b_1 + \alpha_1\alpha_0 b_1 b_2 - \alpha_1 b_2 + 1, -\alpha_2 c_1 + \alpha_3\alpha_2 c_1 c_2 - \alpha_3 c_2 + 1, \\
-\alpha_4\alpha_5 b_2 c_1^3 + \alpha_4\alpha_5 b_1 c_2 c_1^2 + \alpha_4 b_2 c_1 - \alpha_4\alpha_5 b_2 c_2^2 c_1 - \alpha_4 b_1 c_2 + \alpha_4\alpha_5 b_1 c_2^3 + \\
2\alpha_4\alpha_5 b_2 c_1^2 d_1 - \alpha_4\alpha_5 b_1 c_1^2 d_2 - \alpha_4\alpha_5 b_2 c_1 d_1^2 - \alpha_4\alpha_5 b_1 c_2 c_1 d_1 + \alpha_4\alpha_5 b_2 c_2 c_1 d_2 + \\
\alpha_4\alpha_5 b_1 c_1 d_1 d_2 + \alpha_4\alpha_5 b_1 c_2 d_2^2 + \alpha_4\alpha_5 b_2 c_2^2 d_1 - 2\alpha_4\alpha_5 b_1 c_2^2 d_2 - \alpha_4\alpha_5 b_2 c_2 d_1 d_2 - \alpha_4 b_2 d_1 + \\
\alpha_4 b_1 d_2 - \alpha_5 c_1^2 - \alpha_5 c_2^2 + \alpha_5 c_1 d_1 + \alpha_5 c_2 d_2 + 1, \\
2b_2 c_1^2 d_1 - b_1 c_1^2 d_2 - b_2 c_1 d_1^2 - b_1 c_2 c_1 d_1 + b_2 c_2 c_1 d_2 + b_1 c_1 d_1 d_2 + b_1 c_2 d_2^2 + b_2 c_2^2 d_1 - \\
2b_1 c_2^2 d_2 - b_2 c_2 d_1 d_2 - b_2 c_1^3 + b_1 c_2 c_1^2 - b_2 c_2^2 c_1 + b_1 c_2^3, \\
b_1 c_1 + b_2 c_2\},
\end{gathered}
$$

whose Gröbner basis is again $\{1\}$, hence, the statement is proved.