Exercise Sheet 1

To be submitted by email to manuel@kauers.de until **27.05.2018**

You are encouraged to use computer algebra systems for solving the exercises below. You may submit a transcript of your session as (part of) your solution.

Task 1 a) Show that there is a unique formal power series $f \in \mathbb{Q}[[x]]$ with

$$2xf(x) + \mathrm{e}^x(x+1)f(x)^2 + (2x-1)f'(x) = 0$$

and $f(x) = 1 + x + 4x^2 + \frac{65}{6}x^3 + \cdots$.

b) Show that the series f from part a) is D-finite.

Task 2 Show that if the sequence of prime numbers is D-finite, then any recurrence it satisfies has order ≥ 10 or degree ≥ 50.

Bonus problem (not required): can you show that this sequence is not D-finite?

Task 3 Consider a recurrence of order r and degree d with a leading coefficient polynomial whose largest integer root is n_0. The algorithm presented in the lecture for finding a basis of the solution space in $C^{\mathbb{N}}$ of such a recurrence requires $\mathrm{O}((n_0+r)^3)$ operations in C. That's not very good when n_0 is very large. Show that the task can also be done using only $\mathrm{O}(drn_0 + dr^3)$ operations in C.

Task 4 Determine a basis of the solution space in $C[[[x]]]$ of the differential equation

$$\begin{aligned}&x^2(x^2-2)(2x+1)^2 f''(x)\\&- x(16x^3+9x^2-24x-14)(2x+1)f'(x)\\&+ (80x^4+91x^3-65x^2-106x-30)f(x) = 0.\end{aligned}$$

It suffices to find the first five nonzero terms of each basis element.