Now consider the case when there are irregular singularities.

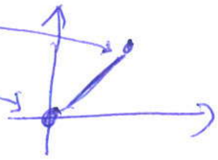
<u>Recall:</u> When the assoc rec of the deq $p_0 f + \dots + p_r f^{(r)} = 0$ is $q_0(n) a_n + \dots + q_s(n) a_{n+s} = 0$ then we can find $\deg q_s$ many l.i. sols of the deq in $C[[x]] = \sum_{\alpha \in C} x^\alpha C[[x]][\log x]$

When $\deg q_s = r$, this is the end of the story (regular case). Plan for today: determine the missing sols when $\deg q_s < r$.

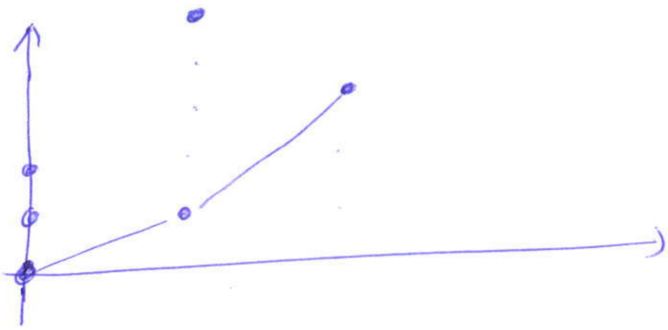
Consider the <u>Newton polygon</u> of the deq. It is defined as (the lower part of) the convex hull of all points $(i,j) \in \mathbb{N}^2$ such that $[x^j] p_i \neq 0$.

Ex:

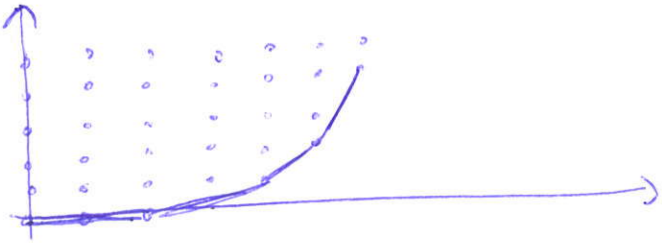
(1) $x^2 y' + 1y = 0$



(2) $x^3 y'' + x(x^4+1)y' + (1+x+x^2)y = 0$



(3)



Note: Sols in $\mathbb{C}[[x]]$ come from edges with slope ≤ 1. This is essentially because $(x^\alpha)' = \alpha x^{\alpha-1}$, so

$$p_{ij}\, x^j (x^\alpha + \dots)^{(i)} = p_{ij}\, \alpha^{\underline{i}}\, x^{\alpha+j-i} + \dots$$

$$p_{i+1,j+1}\, x^{j+1} (x^\alpha + \dots)^{(i+1)} = \underbrace{p_{i+1,j+1}\, \alpha^{\underline{i+1}}\, x^{\alpha+j-i}}_{\substack{\text{cancellation is}\\ \text{possible if we}\\ \text{choose } \alpha \text{ properly.}}} + \dots$$

On the other hand, $x^2y' + y = 0$ has no chance to have a sol in $C[[x]][x]$ because

$$\left.\begin{array}{l} x^2 \cdot (x^\alpha + \dots)' = \alpha x^{\alpha+1} + \dots \\ 1 \cdot (x^\alpha + \dots) = 1 \cdot x^{\alpha+0} + \dots \end{array}\right\} \text{mismatch of exponents}$$

Nontrivial possibilities for α require that the lowest order terms of $p_{i_1}(x)(x^\alpha + \dots)^{(i_1)}$ and $p_{i_2}(x)(x^\alpha + \dots)^{(i_2)}$ for some $i_1 \neq i_2$ cancel and no $p_i(x)(x^\alpha + \dots)^{(i)}$ produces a term of even lower order.

<u>Idea:</u> Transform the deq so that edges with slope > 1 become more flat.

We will exploit

$$(e^{cx^q})' = cq\, x^{q-1} e^{cx^q}$$

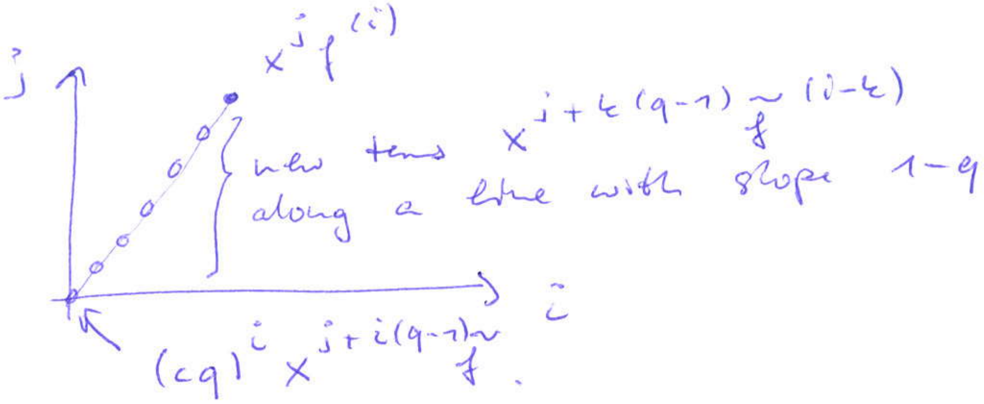
$$(e^{cx^q})'' = (cq)^2 x^{2(q-1)} e^{cx^q} (1 + \dots)$$

$$\vdots$$

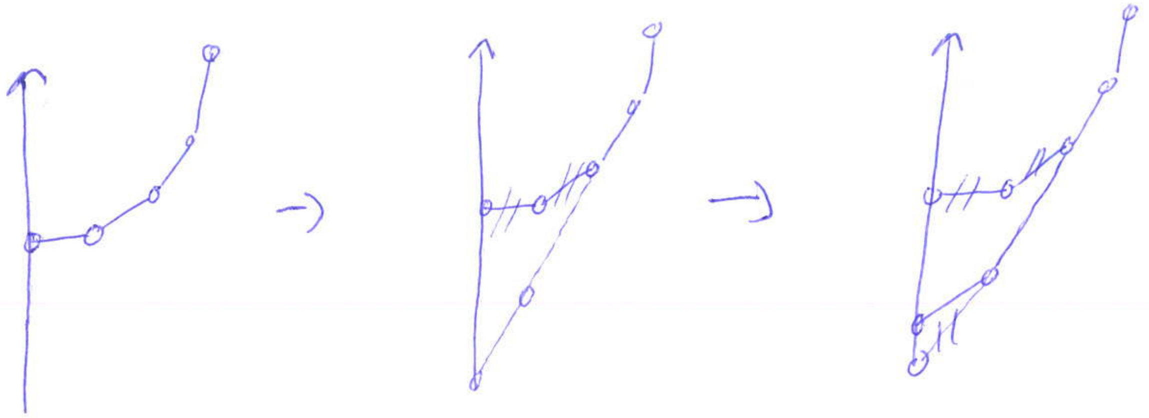
$$(e^{cx^q})^{(i)} = (cq)^i x^{i(q-1)} e^{cx^q} (1 + \dots)$$

So setting $f = e^{cx^q}\tilde{f}$ into the deq and afterwards dividing out e^{cx^q}, every term $x^j f^{(i)}$ will get replaced by

$$\sum_{k=0}^{i} \binom{i}{k} (cq)^k x^{j+k(q-1)} \tilde{f}^{(i-k)}$$



By choosing $q \in \mathbb{Q}$ such that $1-q$ is the slope of an edge in the Newton polygon, we get at least two contributions to the lowest order term of the coeff of $\tilde{f}$ in the new deq. Assuming that C is alg closed, there is a nonzero choice $c \in C$ to make this term vanish. The result is a new Newton polygon with a flatter edge.



The deq for $\tilde{f}$ can be solved recursively. Ensuring that during the recursion we only consider edges with smaller slope, it can be shown that the recursion terminates and produces a full set of r C-l.i. sols of the form

$$\exp(c_1 x^{q_1} + \dots + c_m x^{q_m})\, x^{\alpha}\, a(x^{1/s}, \log x)$$

where $q_1, \dots, q_m \in \mathbb{Q}$, with ~~$0 \leq q_i < 1$~~ $q_i < 0$, $c_1 \dots c_m \in C$, $\alpha \in C$, $s = \operatorname{lcm}(\operatorname{denoms}(q_1, \dots, q_m))$, $a \in C[[x]][y]$.