Recall: 0 is a non-apparent singularity
$\Leftrightarrow$ the deq does not have a full set of solutions in $\mathbb{C}[\![x]\!]$

How can we extend $\mathbb{C}[\![x]\!]$ to a larger domain in which there is always a full set of solutions? (Compare the construction of Puiseux series for solutions of algebraic equations)

Ex:
(1) $xy' - \alpha y = 0$ has solution x^α
(2) $x^2y' + y = 0$ has solution $e^{1/x}$
(3) $xy'' + y' = 0$ has solution $\log(x)$

In a sense, these examples illustrate all that can happen. More precisely we have the following theorem:

Thm: Suppose that C is algebraically closed, let $p_0, \dots, p_r \in C[[x^{1/N}]]$ Puiseux series, $p_r \neq 0$. Then there exist r linearly independent "series" of the form

$$f = \exp(Q(x^{-1/s}))\, x^{\alpha}\, a(x^{1/s}, \log(x))$$

with $Q \in C[x]$, $s \in \mathbb{N}$, $\alpha \in C$, $a \in C[[x]][y]$ such that $p_0 f + \dots + p_r f^{(r)} = 0$.

Note: We view $\log(x)$, x^{α}, etc not as functions but as formal objects on which differentiation is defined as suggested by the notation, e.g. $(x^{\alpha})' = \alpha x^{\alpha-1}$ and $\log(x)' = \frac{1}{x}$. We also assume relations like $x^{\alpha+\beta} = x^{\alpha} x^{\beta}$ $(\alpha, \beta \in C)$.

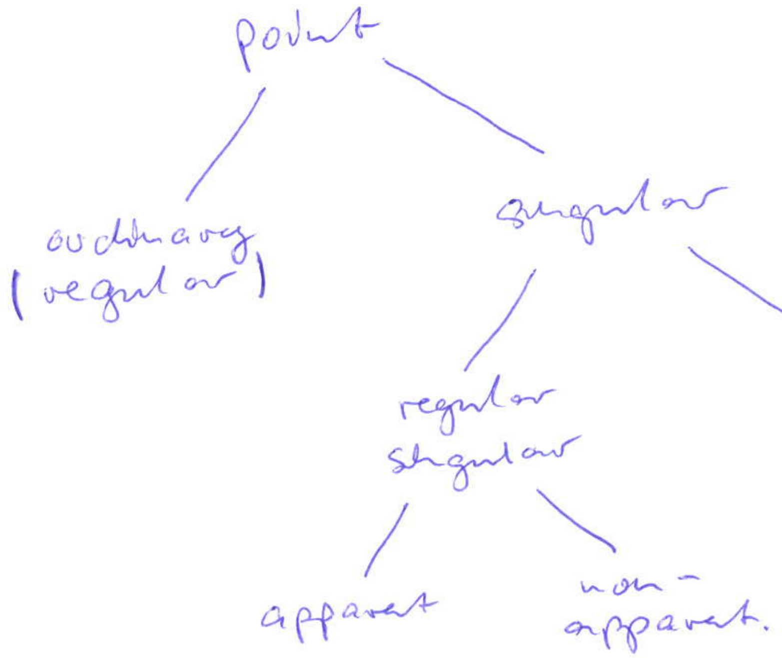
Note that substituting f into the ~~de~~ rhs of the deq gives a "series" of the same form: Q, s, α agree (if $r=1$).

<u>Def</u>: Write $\mathbb{C}[\![x]\!]$ for the set of all $\mathbb{C}$-linear-combinations of series in

$$\bigcup_{\alpha\in\mathbb{C}} x^{\alpha}\,\mathbb{C}[\![x]\!][\log(x)].$$

Observe that $\mathbb{C}[\![x]\!]$ is a differential ring and an integral domain.

<u>Def</u>: A singularity ξ of a deq is called <u>regular</u> if the deq has a full set of solutions in $\mathbb{C}[\![x]\!]$. Otherwise it is called irregular.



Plan for today: how to find solutions in $\mathbb{C}[\![x]\!]$.

Recall: Every deq $p_0 f + \dots + p_r f^{(r)} = 0$ can be translated into a rec $q_0(n)a_n + \dots + q_s(n)a_{n+s} = 0$ for the series coeffs of a solution $f = \sum_n a_n x^n$

Remarks:

(1) For this translation, we did not need to assume that n runs through the integers. A sequence solution with support in $\alpha + \mathbb{N}$ can be interpreted as a series $\sum_{n=0}^{\infty} a_{n+\alpha} x^{n+\alpha} = x^\alpha \sum_{n=0}^{\infty} a_{n+\alpha} x^n$.

(2) Although not every solution of the rec corresponds to a series solution of the deq, it can be checked that every solution $a: \alpha + \mathbb{Z} \to \mathbb{C}$ with $\exists n_0 \in \mathbb{Z}\ \forall n < n_0: a_{n+\alpha} = 0$ does give rise to a series solution.

Consequences:

(1) If $x^\alpha \sum_{n=0}^{\infty} a_n x^n$ with $a_0 \neq 0$ is a solution of a deq, then α must be a root of $q_s(x-s)$. This polynomial is called the indicial polynomial of the deq.

(2) If q_s is squarefree and $\deg q_s = r$ (and C is algebraically closed) and for any two roots α_1, α_2 of q_s we have $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, then we get r linearly independent sols in $C[[x]]$. They don't have logs.

Multiple roots and roots with integer distance (may) lead to solutions involving logs. To see how, make an ansatz for a solution of the form

$$f(x) = x^u \sum_{n=0}^{\infty} a_n x^n$$

where u is a new variable and $a_0 = 1$.

Plugging f into the deq and comparing coeffs wrt x gives the perturbed rec

$$q_0(n+u)a_n + \dots + q_s(n+u)a_{n+s} = 0$$

As long as u is a variable, this rec can't have any sols in $C(u)^{\mathbb{Z}}$ which are zero for all sufficiently small n, because $q_s(x+u)$ has no roots in C, hence also no roots in $\mathbb{Z}$, hence there are no candidates for starting points. (Note: $q_s(\xi+u)=0 \iff \xi-u$ is a root of $q_s \in C[x]$).

Define $(a_n) \in C(u)^{\mathbb{Z}}$ by setting $a_n = 0$ $\forall n<0$, $a_0 = 1$, and a_n for $n>0$ in accordance with the rec. Then the rec will only be violated for $n=-s$. For the corresponding series $f(x) = x^u \sum_{n=0}^{\infty} a_n x^n$ we have

$$p_0 f + \dots + p_r f^{(r)} = q_s(u-s)\, x^{u-s}$$

It is tempting to set u to a root α of $q_s(x-s)$ to get a solution of the deq. But not too fast! What if some of the $a_n \in \mathbb{C}(u)$ have powers of $u-\alpha$ in their denominator? Such factors can get introduced by unrolling the rec

$$a_n = -\frac{1}{q_s(n+u-s)}\left(q_0(n+u-s)a_{n-s} + \dots + q_{s-1}(n+u-s+1) \cdot a_{n-1}\right)$$

when n is such that $q_s(n-\alpha-s)=0$. Since q_s is a polynomial, there can be at most finitely many such n.

Therefore, for every root α of $q_s(x-s)$ there exists $e \in \mathbb{N}$ such that it's legal to evaluate $((u-\alpha)^e a_n)_{n\in\mathbb{Z}}$ at $u=\alpha$. Choose the smallest such e, so that setting $u=\alpha$ does not give the zero sequence.

Plugging $(u-\alpha)^e f(x)$ into the lhs of the deq gives

$$q_s(u-s)\cdot(u-\alpha)^e x^{u-s}.$$

Let k be the multiplicity of α as a root of $q_s(x-s)$. Then we have

$$\left[\frac{d^i}{du^i}\left(q_s(u-s)(u-\alpha)^e\right)\right]_{u=\alpha}=0$$

for $i=0\ldots e+k$, so the series

$$\left[\frac{d^i}{du^i}(u-\alpha)^e f(x)\right]_{u=\alpha}$$

for $i=0\ldots e+k$ are sols of the deq.

Note:

(1) For $i<e$ we have $[x^\alpha]\left(\frac{d^i}{du^i}(u-\alpha)^e f(x)\right)_{u=\alpha}=0$ but for $i\geq e$, this coeff is $\neq 0$.

(2) $\frac{d}{du}x^u=\frac{d}{du}e^{u\log x}=(\log x)x^u$, so this is where the log terms come from.

(3) Multiple roots of q_s always lead to logs ($k>0$), but roots at integer distance may or may not lead to logs (depending on whether $e>0$).

(4) For each root α of $q_s(x-s) \in \mathbb{C}[x]$ we obtain as many sols in $\mathbb{C}[[\,]] \times [[\,]]$ as its multiplicity. Taking all roots we get altogether $\deg q_s$ many linearly independent sols in $\mathbb{C}[[\,]] \times [[\,]]$.

In particular: 0 is a regular singular point $\Leftrightarrow \deg q_s = r$.

(strictly speaking, we have only shown "$\Leftarrow$", but "$\Rightarrow$" follows from the fact that the described procedure actually finds all solutions.)