There is a divide and conquer variant of the algorithm with cost $O(r\,M(rN)\log N)$ where $M(n)$ is the time to multiply two polynomials in $C[x]$ of degree $\leq n$.
(Fact: $M(n) = O(n \log n \log\log n)$)
With $N \approx rd$ and $d \approx r^2$, this is just $O(r^5 \log r)$ operations in C. (world record)

5 Singularities

Recall:

(1) A deq $p_0 y + \cdots + p_r y^{(r)} = 0$ with $p_0 \ldots p_r \in C[x]$ and $x \nmid p_r$ has a solution space of dimension r in $C[[x]]$.

(2) A rec $p_0(n) a_n + \cdots + p_r(n) a_{n+r} = 0$ with $p_0 \ldots p_r \in C[x]$ and $\mathrm{roots}(p_r) \cap \mathbb{Z} = \emptyset$ has a solution space of dimension r in $C^{\mathbb{N}}$ (or in $C^{\mathbb{Z}}$)

<u>Question:</u> What happens in the other cases?

<u>Def:</u>

(1) A root of $p_r \neq 0$ is called a singularity of the deq $p_0 y + \dots + p_r y^{(r)} = 0$ with $p_0 \dots p_r \in \mathbb{C}[x]$.

(2) An equivalence class $[\xi] \in \mathbb{C}/\mathbb{Z}$ is called a singularity of the rec $p_0(n) a_n + \dots + p_r(n) a_{n+r} = 0$ with $p_0 \dots p_r \in \mathbb{C}[x]$ and $p_r \neq 0$ iff it contains a root of p_r.

<u>Note:</u> By applying suitable changes of variables, it suffices to understand the cases 0 and $[0]$.

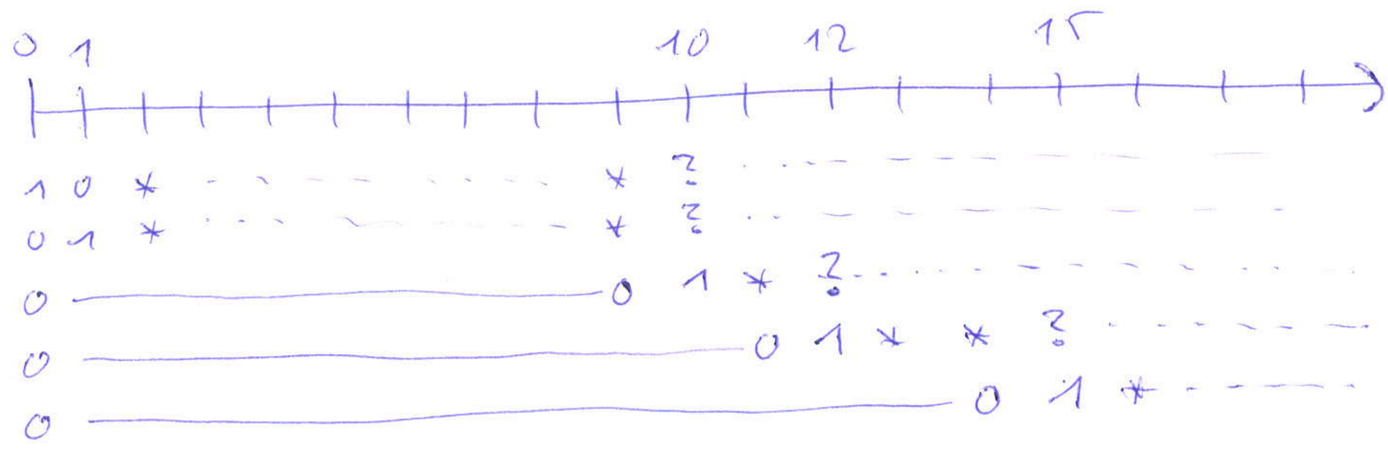
<u>Ex:</u> Consider a rec of order 3 with $p_3 = (x-7)(x-9)(x-12)$. What can its solution space look like?

$$(n-7)(n-9)(n-12)\,a_{n+3} = \bigcirc a_n + \bigcirc a_{n+1} + \bigcirc a_{n+2}$$

For $n = 7, 9, 12$, the lhs is zero regardless of the value of a_{n+3}.

<u>Case 1</u>: If the earlier values are such that the rhs is not zero, then there is <u>no</u> valid continuation for these values which is consistent with the rec.

<u>Case 2</u>: If the earlier values are such that the rhs is zero, then <u>any</u> choice of a_{n+3} is a valid continuation which is consistent with the rec.



* ... uniquely determined value

? ... may or may not exist

Observations:

(1) If n_0 is a root of p_r, then a new solution is "born" at index $n_0 + r$ (it is zero before). Some of the older solutions may "die" at this point.

(2) The picture above is misleading: it may be that two basis solutions die but a certain linear combination of them survives.

(3) No solution can die past the largest integer root of p_r (plus r).

Alg:

Input: $p_0 \dots p_r \in \mathbb{C}[x]$, $p_r \neq 0$.

Output: a basis of the solution space in $\mathbb{C}^{\mathbb{N}}$ of the rec $p_0(n) a_n + \dots + p_r(n) a_{n+r} = 0$.

The basis elements are given in terms of finite prefixes which have unique extensions to infinite sequence solutions.

(1) Find $n_0 \in \mathbb{N}$ such that $p_r(n) \neq 0$ for all $n > n_0$

(2) Make an ansatz with undetermined coefficients $a_0, \dots, a_{n_0+r}$. Evaluate the rec for $n = 0 \dots n_0$ to obtain a linear system for the variables.

(3) solve the system and return a basis of the solution space.

Correctness follows from the observations made above.

<u>Thm:</u> The solution space in $C^{\mathbb{N}}$ or $C^{\mathbb{Z}}$ of a rec $p_0(n) a_n + \dots + p_r(n) a_{n+r} = 0$ $p_0 \dots p_r \in C[x]$, $p_r \neq 0$ has dimension $\geq r$ and $\leq r + m$, where m is the number of distinct integer roots of p_r.

Proof: $\geq$ follows from the Alg because the linear system has size $(n_0+1)\times(n_0+1+r)$. $\leq r+m$ follows from the observation that new solutions can only be born at integer roots of p_r ∎

Note: The theorem does not contradict our earlier result according to which the dimension is always bounded by the order. That result is only valid for solution spaces in difference fields, while sequences don't form a field.

Variation: Sometimes we care about sols (a_n) in $C^{\mathbb{Z}}$ coeff $\exists n_0 \forall n < n_0 : a_n = 0$. Such sols can only start to be nonzero at a root of p_r (plus r). The space of such sols has dim bounded by m.

<u>Differential case:</u> Here we know that $\dim V \leq r$ for the sol space $V \subseteq \mathbb{C}[[x]]$ because $\mathbb{C}[[x]]$ is an integral domain (so it's contained in a field).

If ξ is a regular point (i.e. not a singularity) then all sols of the deq are smooth at ξ.

If a solution of a deq has a singularity (e.g. a pole) at a point ξ, then ξ must be a singularity of the deq.

<u>Ex:</u>

$y' - y = 0 \quad \rightarrow \quad y = c \cdot \exp(x)$

no singularity at 0

$x y' + y = 0 \quad \rightarrow \quad y = c \cdot \frac{1}{x}$

singularity at 0

(but nowhere else)

But:

$xy' - 5y = 0 \longrightarrow y = cx^5$

no singularity

"false alarm".

<u>Def:</u> A singularity ξ of a deq

$p_0 y + \dots + p_r y^{(r)} = 0 \quad (p_0, \dots, p_r \in C[x],\ p_r \neq 0)$

is called <u>apparent</u> if the solution space of the deq in $C[[x-\xi]]$ has dimension r.

This can be checked as follows:

<u>Alg:</u>

Input: $p_0, \dots, p_r \in C[x],\ p_r \neq 0$

Output: A basis of the sol space in $C[[x]]$ of $p_0 y + \dots + p_r y^{(r)} = 0$. Each basis element is given by sufficiently many terms to allow for a unique continuation.

(1) Compute the rec associated to the deq

(2) find $n_0 \in \mathbb{N}$ such that the leading coeff poly of the rec is not zero for any $n \geq n_0$

(3) Make an ansatz $y = \sum_{n=0}^{n_0+d+2r} a_n x^n$ with unknown coeffs $a_0, a_1, \ldots$

(4) equate the coeffs of $p_0 y + \cdots + p_r y^{(r)}$ mod x^{n_0+d+r} wrt x to zero and solve the resulting linear system.

(5) return a basis of the solution space.

<u>Thm:</u> Let $p_0, \ldots, p_r \in \mathbb{C}[x]$, $p_r \neq 0$, $\gcd(p_0, \ldots, p_r) = 1$. Then:

0 is a singularity of $p_0 y + \cdots + p_r y^{(r)} = 0$ if and only if the deq does not have a solution space with a basis of the form

$$1 + 0x + \dots + 0x^{r-1} + \dots$$
$$0 + 1x + \dots + 0x^{r-1} + \dots$$
$$\vdots$$
$$0 + 0x + \dots + 1x^{r-1} + \dots$$

Proof:
"$\Leftarrow$" was already shown before (when we saw that at a regular point there are always r linearly independent sols in $C[\![x]\!]$; we actually showed that they have the form stated here.)

"$\Rightarrow$" Suppose that for $k = 0 \dots r-1$ there are solutions of the form

$$b_k := x^k + c_k x^r + \dots \in C[\![x]\!].$$

Then $\sum_{i=0}^{r} p_i b_k^{(i)} = 0$, so

$$0 = [x^0] \sum_{i=0}^{r} p_i b_k^{(i)} = \sum_{i=0}^{r} ([x^0]p_i)([x^0]b_k^{(i)})$$

$$= \sum_{i=0}^{r} ([x^0]p_i)\cdot([x^0] \; k^{\underline{i}} x^{k-i} + c_k r^{\underline{i}} x^{r-i} + \dots)$$

$$= \sum_{i=0}^{r} ([x^0]p_i)(k^{\underline{i}} \delta_{i,k} + c_k r^{\underline{i}} \delta_{r,i})$$

$$= k!\, [x^0]p_k + c_k r!\, [x^0]p_r$$

Therefore, if $x \mid p_r$, i.e. $[x^0]p_r = 0$, then also $[x^0]p_k = 0$ for all $k = 0 \dots r-1$, which is impossible by the assumption $\gcd(p_0, \dots, p_r) = 1$. So $x \nmid p_r$, as claimed ☒

As a consequence of the prev. thm, we can desingularize every deq which has a full set of fps sols.