(3) The dimension of V may be less than r.

Ex: (a) $y' - y = 0$, $K = C(x) \Rightarrow V = \{0\}$

(b) $x^2 y' + y = 0$, $K = C((x)) \Rightarrow V = \{0\}$

<u>Thm:</u> Let $p_0, \dots, p_r \in C[x]$, $p_r \neq 0$.

(1) If p_r has no roots in $\mathbb{N}$ then for every choice $\alpha_0, \dots, \alpha_{r-1} \in C$ there is a unique sequence $(a_n)_{n=0}^{\infty}$ with $a_n = \alpha_n$ for $n = 0 \dots r-1$ and

$$p_0(n) a_n + \dots + p_r(n) a_{n+r} = 0$$

for all $n \in \mathbb{N}$.

(2) If $x \nmid p_r$ then for every choice $\alpha_0, \dots \alpha_{r-1} \in C$ there is a unique fps $a(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_n = \alpha_n$ for $n = 0 \dots r-1$ and

$$p_0(x) a(x) + \dots + p_r(x) a^{(r)}(x) = 0.$$

Proof:

(1) clear by induction:

$$a_n = -\frac{1}{\underbrace{\boxed{p_r(n-r)}}_{\neq 0 \text{ for } n \geq r}} \left(p_0(n) a_{n-r} + \dots + p_{r-1}(n) a_{n-1} \right)$$

(2) Also by induction. Suppose that $a(x) \equiv \sum_{n=0}^{N-1} a_n x^n$ is such that

$$p_0 a + \dots + p_r a^{(r)} = 0 \mod x^{N-r}$$

and make an ansatz $\tilde{a} = a + a_N x^N$ with undetermined a_N. Then

$$p_0 \tilde{a} + \dots + p_r \tilde{a}^{(r)}$$
$$= \Big(\underbrace{\boxed{}}_{\substack{\text{a certain } \mathbb{C}\text{-linear combination of} \\ a_{n-1},\, a_{n-2},\, \dots}} + p_r(0) N^{\underline{r}} a_N \Big) x^{N-r} \mod x^{N-r+1}$$

Because of $p_r(0) \neq 0$ ($\Leftrightarrow x \nmid p_r$) and $N^{\underline{r}} \neq 0$ ($\Leftarrow N \geq r$), there is one and only one choice for a_N to have the coefficient of x^{N-r} become zero. ☒

<u>Thm:</u> $(a_n)_{n=0}^{\infty} \in C^{\mathbb{N}}$ is D-finite as sequence

$\Leftrightarrow$ $a(x) = \sum_{n=0}^{\infty} a_n x^n \in C^{\mathbb{N}}$ is D-finite as f.p.s.

More precisely:

(1) If $a(x)$ satisfies a deq of order $\leq r$ and degree $\leq d$ (viz with polynomial coefficients of degree $\leq d$) then $(a_n)_{n=0}^{\infty}$ satisfies a recurrence of order $\leq r+d$ and degree $\leq r$.

(2) If $(a_n)_{n=0}^{\infty}$ satisfies a rec of order $\leq r$ and degree $\leq r$, then $a(x)$ satisfies a deq of order $\leq d$ and degree $\leq r+d$?

<u>Proof:</u>

(1) We have $a^{(i)}(x) = \sum_{n=0}^{\infty} n^{\underline{i}} a_n x^{n-i} = \sum_{n=0}^{\infty} n^{\underline{i}} a_{n+i} x^n$

for $i \in \mathbb{N}$.

Write $p_i = \sum_{j=0}^{d} p_{ij} x^j$. Then

$$0 = p_0 a + \dots + p_r a^{(r)}$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{d} p_{ij} x^j \sum_{n=0}^{\infty} n^{\underline{i}} a_{n+i} x^n$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{d} \sum_{n=j}^{\infty} p_{ij} (n-j)^{\underline{i}} a_{n-j+i} x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{r} \sum_{j=0}^{\min(n,d)} p_{ij} (n-j)^{\overline{i}} a_{n-j+i} \right) x^n$$

$$\Rightarrow \forall n \geq d: \sum_{i=0}^{r} \sum_{j=0}^{d} p_{ij} (n-j)^{\overline{i}} a_{n-j+i} = 0$$

$$\Rightarrow \forall n \geq 0: \underbrace{\sum_{i=0}^{r} \sum_{j=0}^{d} p_{ij} (n+d-j)^{\overline{i}} a_{n+d-j+i}}_{= \sum_{j=0}^{d} p_{i,d-j} (n+j)^{\overline{i}} a_{n+i+j}} = 0$$

This is the announced recurrence of order $\leq r+d$ and degree $\leq r$.

(2) Similarly. ☒

<u>Ex:</u>

(1) $y' - y = 0 \iff (n+1) a_{n+1} - a_n = 0$

(2) $x y' - 5y = 0 \iff (n-5) a_n = 0$

(3) $(1-x) y' + y = 0 \iff a_{n+1} - a_n = 0$

Note:

(1) When the leading coeff poly of a recurrence has integer roots, it can have at most finitely many of them, so one of them will be the largest. Beyond this point, the recurrence uniquely determines all the subsequent terms. The values before the largest integer root are finitely many and can be tabulated. So every D-finite sequence can be described by a finite amount of information (rec + init vals).

(2) Via the previous thm, also every D-finite fps can be described by a finite amount of information (deg + init vals).

Thm:

(1) If $a, b \in C[[x]]$ are D-finite, then so are $a+b$ and $a \cdot b$.

(2) If $(a_n), (b_n) \in C^{\mathbb{N}}$ are D-finite, then so are $(a_n + b_n)$ and $(a_n b_n)$.

Proof (for (2) and times). Suppose that

$$p_0(n) a_n + \dots + p_r(n) a_{n+r} = 0 \quad \forall n \in \mathbb{N}$$
$$q_0(n) b_n + \dots + q_s(n) b_{n+s} = 0 \quad \forall n \in \mathbb{N}$$

for some polynomials $p_i, q_i \in C[x]$. Because of the recurrences, for every $i \in \mathbb{N}$ there are rational functions $u_0, \dots, u_{r-1} \in C(x)$ such that

$$a_{n+i} = u_0(n) a_n + \dots + u_{r-1}(n) a_{n+r-1}$$

for all (sufficiently large) $n \in \mathbb{N}$, likewise with b_{n+i}. Then for all $i, j \in \mathbb{N}$ there are rational functions $w_{k,e} \in C(x)$ such that

$$a_{n+i} b_{n+j} = \sum_{e=0}^{s-1} \sum_{k=0}^{r-1} w_{k,e}(n) \underbrace{a_{n+e} b_{n+k}}_{r \cdot s \text{ many terms}}$$

By linear algebra, it follows that $a_n b_n, \dots, a_{n+rs} b_{n+rs}$ are linearly dependent over $\mathbb{C}(x)$ ▨

The proofs for the other claims are similar.

<u>Thm:</u> If $(a_n), (b_n) \in \mathbb{C}^{\mathbb{N}}$ are D-finite, then so is $\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$. (Cauchy-Product)

<u>Proof:</u>

$(a_n), (b_n)$ D-finite

$\Rightarrow a(x) = \sum_{n=0}^{\infty} a_n x^n,\ b(x) = \sum_{n=0}^{\infty} b_n x^n$ D-finite

$\Rightarrow a(x) b(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$ D-finite

$\Rightarrow \left(\sum_{k=0}^{n} a_k b_{n-k}\right)$ D-finite. ▨

Analogously, $a(x) \odot b(x) := \sum_{n=0}^{\infty} a_n b_n x^n$ is D-finite when $a(x), b(x) \in \mathbb{C}[[x]]$ are. (Hadamard-product)

Operations which preserve D-finiteness are known as "closure properties".

Here are some more:

Thm:

(1) If (a_n) is D-finite then so is $(a_{\lfloor un+v \rfloor})$ for any fixed positive $u, v \in \mathbb{Q}$.

(2) If $(a_n), (b_n)$ are D-finite, then so is their interlacing $a_0 b_0 a_1 b_1 a_2 b_2 \ldots$

(3) If $a(x)$ is D-finite and $b(x)$ is algebraic, then $a(b(x))$ is D-finite.

(4) If $a(x)$ is D-finite then so is $\int a(x)dx$.

Proof idea is always the same: set up a linear system for the coeffs of the required equation and observe that it is underdetermined, so that a solution exists by linear algebra.

Using closure properties it is often easy to see that something is D-finite.

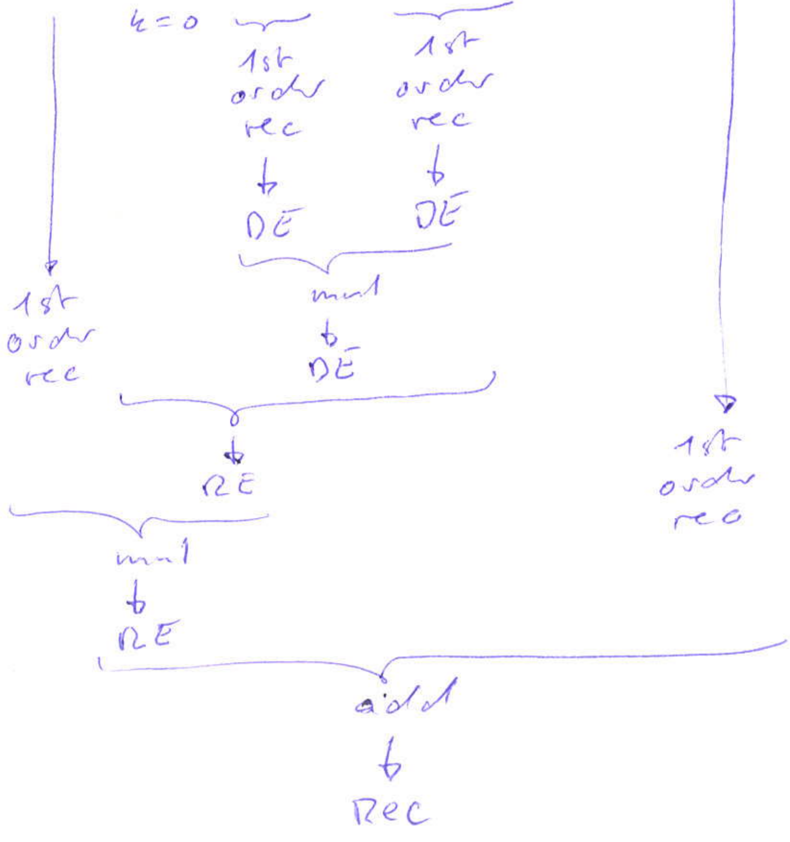
Ex: (1) $\int e^{\sqrt{1-x^2}} dx + \dfrac{x^2 \log(x)}{2 + \sqrt[3]{x}}$.

(2) $\left(\sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{k+1}\right)^2 - 5 \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{2^i - 3^i}{5i+3}$.

Closure properties are also useful for proving identities:

Ex: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$$n! \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{(n-k)!} - 2^n$$



⇝ check sufficiently many initial terms to ensure that this is identically zero.

It is usually much harder to see that something is not D-finite. The three main tools for doing so are:

(a) the asymptotics is not right.
Ex: 2^{n^2} grows too fast to be D-finite
(we will see later which growths are possible)

(b) there are too many singularities
Ex: $\tan x$ has infinitely many and can therefore not be D-finite.
(we will see later that singularities can only appear at roots of the leading coeff poly, so there are at most finitely many)

(c) Thm (w/o proof)
(1) f and $\frac{1}{f}$ are both D-finite
$\iff$ $\frac{f'}{f}$ is algebraic
(2) (a_n) and $(\frac{1}{a_n})$ are both D-finite
$\iff$ (a_n) is an interlacing of finitely many D-finite seqs with a rec of order 1.

Ex: $a(x) = \frac{x}{1-e^x}$ is not D-finite because

$\frac{1}{a(x)} = \frac{1-e^x}{x}$ is and $\frac{a'(x)}{a(x)} = + \frac{e^x x - e^x + 1}{x(1-e^x)}$

is not.