Now let (K, D) be an arbitrary differential field, and $m \in K[Y]$ irreducible, so that $K(y) := K[Y]/\langle m \rangle$ is a field. Recall that $K(y)$ is a K-vector space of dimension $\deg_Y m$, with $1, y, \dots, y^{\deg_Y m - 1}$ as a basis.

<u>Thm:</u> There is exactly one derivation $\bar{D} : K(y) \to K(y)$ with $\bar{D}|_K = D$.

<u>Proof:</u> For any choice $p \in K[Y]$ there is a unique derivation $\bar{D} : K[Y] \to K[Y]$ with $\bar{D}(Y) = p$. In order to get a well-defined derivation on the quotient ring $K[Y]/\langle m \rangle$ it is necessary and sufficient to choose p such that $\langle m \rangle$ is closed under $\bar{D}$, i.e. such that $\bar{D}(m(y)) = 0$. Writing $m = m_0 + m_1 Y + \dots + m_r Y^r$, this forces

$$0 = D(m_0) + D(m_1)y + \dots + D(m_r)y^r$$
$$+ m_1 \bar{D}(y) \qquad\qquad + m_r r y^{r-1} \cdot \bar{D}(y)$$

$$= \underbrace{D(m)}_{\substack{\text{coefficient} \\ \text{wise derivative} \\ \text{of } m}} + \bar{D}(y) \cdot \underbrace{(D_Y m)(y)}_{\substack{\text{usual derivative} \\ \text{of } m \text{ wrt } y.}}$$

so we must have $\bar{D}(y) = -\frac{D(m)(y)}{(D_y m)(y)}$.

(Note that $(D_y m)(y) \neq 0$ because m is irreducible.)

Hence there is exactly one choice for $\bar{D}$ ☒

Ex: $m = Y^2 - x$

$0 = \bar{D}(y^2 - x) = 2y\bar{D}(y) - 1$

$\Rightarrow \quad \bar{D}(y) = \frac{1}{2y} = \frac{1}{2}\frac{y}{y^2} = \frac{1}{2x}y.$

Thm (Abel) Every algebraic function is D-finite.

Proof. Let y be an algebraic function with minimal polynomial $m \in C(x)[Y]$. Then $C(x)[Y]/\langle m \rangle$ is a differential field by the previous thm. It is also a $C(x)$-vector space of dimension $r = \deg_Y m$, so any choice of $r+1$ many elements will be linearly dependent over $C(x)$. In particular, there exist $p_0, \dots, p_r \in C(x)$, not all zero, such that

$$p_0 y + p_1 D(y) + \dots + p_r D^r(y) = 0.$$

☒

$$
\begin{aligned}
y &= y \\
D(y) &= \ast + \ast\, y + \cdots + \ast\, y^{r-1} \\
D^2(y) &= \ast + \ast\, y + \cdots + \ast\, y^{r-1} \\
&\;\vdots \\
D^r(y) &= \ast + \ast\, y + \cdots + \ast\, y^{r-1}
\end{aligned}
$$

$$
\begin{array}{c c}
 & \begin{array}{ccc} y & D(y) \cdots & D^r(y) \end{array} \\
\begin{array}{c} 1 \\ y \\ \vdots \\ y^{r-1} \end{array} & \underbrace{\begin{pmatrix} \ast & \cdots & \ast \\ \vdots & \ddots & \vdots \\ \ast & \cdots & \ast \end{pmatrix}}_{\text{more variables than equations}}
\end{array}
\cdot \begin{pmatrix} p_0 \\ \vdots \\ p_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
$$

$\Rightarrow$ nontrivial solution.

Note that the proof also shows that there is always a differential equation whose order is at most the degree of the minimal polynomial. It may be smaller, though, eg $y^2 - x = 0 \Rightarrow 2x\,D(x) - 1 = 0$.

<u>Thm:</u> If y_1, y_2 are algebraic functions then so are $D(y_1)$, $y_1 + y_2$, $y_1 \cdot y_2$, $y_1 \circ y_2$, $\frac{1}{y_1}$, y_1^{-1} (functional inverse).

<u>Proof</u> (for + and $\cdot$) Let $m_1, m_2 \in C(x)[Y]$ be the minimal polynomials of y_1, y_2, respectively, and consider the differential field $C(x)[Y_1, Y_2]/\langle m_1(Y_1), m_2(Y_2)\rangle$

$$\left(\simeq \big(C(x)[Y_1]/\langle m_1(Y_1)\rangle\big)[Y_2]/\langle m_2(Y_2)\rangle\right).$$

It is a $C(x)$-vector space of dimension $d = \deg m_1 \cdot \deg m_2$. Therefore, the elements

$$1, (y_1+y_2), \ldots, (y_1+y_2)^d$$

are linearly dependent over $C(x)$, likewise for $\cdot$. ∎

<u>Remarks:</u>

(1) Annihilating polynomials for y_1+y_2 etc can be computed by linear algebra, or with Gröbner bases, eg

$$\underbrace{\langle Z-(Y_1+Y_2), m_1(Y_1), m_2(Y_2)\rangle}_{\subseteq C(x)[Z, Y_1, Y_2]} \cap C(x)[Z]$$

(2) If $a_1, a_2 \in C[[x]]$ are algebraic fps with minimal polynomials $m_1, m_2 \in C(x)[Y]$, and m is the annihilating polynomial for $a_1 + a_2$ (or $a_1 \cdot a_2$) computed as described above (linear algebra or Gröbner bases), then m need not be the minimal polynomial. Why not? It's because m does not know which roots of m_1 and m_2 we have in mind, so it has to annihilate all $\tilde{a}_1 + \tilde{a}_2$ where $\tilde{a}_1$ is some solution of m_1 and $\tilde{a}_2$ is some solution of m_2.

(3) If m is not the minimal polynomial of $a_1 + a_2$ (or $a_1 \cdot a_2$), one of its irreducible factors is. If p is an irreducible factor of m, we cannot easily detect that it is the right one, i.e. that $p(x, a_1 + a_2) = 0 \in C[[x]]$,

but we can recognize in a finite number of steps that $p(x, a_1 + a_i) \neq 0$ (if this is the case). So we can find the right factor by excluding the wrong ones.

(4) The set of all ~~computable~~ algebraic fps forms a subring of $\mathbb{C}[[x]]$. This subring is computable. As a representation, we can use the minimal polynomial $m \in \mathbb{C}(x)[Y]$ together with enough initial values to distinguish the series from the other fps solutions of m.

3 D-finite Functions

While an algebraic equation of degree r has (at most) r distinct solutions, the solution set of a linear deq of order r is a vector space over the constant field C. What can we say about its dimension?

Thm: Let K be a differential field with constant field C. Let $p_0, \dots, p_r \in K$ and $V = \{y \in K \mid p_0 y + \dots + p_r D^r(y) = 0\} \subseteq K$. Then V is a C-vector space and $\dim_{\mathbb{C}} V \leq r$.

Proof: It is clear that V is a C-VS, because $D: K \to K$ is a C-linear map. Let $y_0, \dots, y_r \in V$. We show that they are linearly dependent over C.

In fact, we show that the vectors

$\begin{pmatrix} y_i \\ D(y_i) \\ \vdots \\ D^r(y_i) \end{pmatrix}$ $(i = 0 .. r)$ are linearly dependent

over C. They clearly are dependent over K, because the deq implies

$$\begin{vmatrix} y_0 & \cdots & y_r \\ \vdots & & \vdots \\ \vdots & & \vdots \\ D^r(y_0) & \cdots & D^r(y_r) \end{vmatrix} = \begin{vmatrix} y_0 & \cdots & y_r \\ \vdots & & \vdots \\ D^{r-1}(y_0) & \cdots & D^{r-1}(y_r) \\ 0 & \text{———} & 0 \end{vmatrix} = 0.$$

So let $c_0, \dots, c_r \in K$ be such that $\sum_{i=0}^{r} c_i \begin{pmatrix} y_i \\ \vdots \\ D^r(y_i) \end{pmatrix} = 0$. Then, in fact,

$\sum_{i=0}^{r} c_i \, D^j(y_i) = 0$ for all $j \geq 0$ by induction and the deq. Wlog we may assume that

$$(c_0, \dots, c_r) = (*, \dots, *, \underset{\substack{\uparrow \\ k\text{th position}}}{1}, 0 \dots 0)$$

with minimal possible k. Then

$$0 = D\Big(\sum_{i=0}^{r} c_i \begin{pmatrix} y_i \\ \vdots \end{pmatrix}\Big) = \sum_{i=0}^{r} D(c_i) \begin{pmatrix} y_i \\ \vdots \end{pmatrix} + \underbrace{\sum_{i=0}^{r} c_i \begin{pmatrix} D(y_i) \\ \vdots \end{pmatrix}}_{=0}$$

Since $D(c_r) = D(1) = 0$, the minimality of r forces $D(c_i) = 0$ for all i, so $c_0, \dots c_r \in C$, as claimed. ∎

Remarks:

(1) Note that if K is not carefully constructed, Const K may be larger than expected.
Ex: For $K = C(x,y)$ with $D(c) = 0 \; \forall c$, $D(x) = 1$, $D(y) = \frac{2}{x}y$ we have $\frac{y}{x^2} \in \mathrm{Const}\,K$, because $D\left(\frac{y}{x^2}\right) = \frac{D(y)x^2 - yD(x^2)}{x^4} = \frac{2yx - y2x}{x^4} = 0$
We call $\frac{y}{x^2} \in C(x,y) \setminus C$ a "false constant".
We mostly care about the case where K is some extension of $C(x)$ with $D(x) = 1$ and Const $K = C$. See literature on "differential algebra" for a more elaborate discussion.

(2) There is an analogous theorem (with analogous proof) for difference fields and recurrence equations.