2 Algebraic Functions

__Def:__ A fps $f \in \mathbb{C}[[x]]$ is called __algebraic__ if there exists $m \in \mathbb{C}[x,y] \setminus \{0\}$ with $m(x,f) = 0$.

__Ex:__

(1) $f = x^2$ $\qquad m = y - x^2$

(2) $f = \sqrt{1+x}$ $\qquad m = y^2 - (1+x)$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

(3) $m = 2y^2 - (3x+1)y + (4x+5)$

$f = a_0 + a_1 x + \cdots$

$$
\begin{aligned}
& m(x,f) \\
&= 2(a_0^2 + 2a_0a_1 x + \cdots) \\
&\quad - (3x+1)(a_0 + a_1 x + \cdots) \\
&\quad + (4x+5) \\
&= (2a_0^2 - a_0 + 5) + \bullet\, x + \bullet\, x^2 + \cdots
\end{aligned}
$$

So $f \in \mathbb{C}[[x]]$ can only be a solution of $m=0$ if its first coefficient a_0

is a root of $m(0,y) \in C[y]$.

(Note: $f^n = a_0^n + \cdots$)

In general, there cannot be more than $\deg_y m$ many fps solutions. But there may be fewer. Possible issues:

(a) $m(0,y)$ has multiple roots

(b) $\deg_y m(0,y) < \deg_y m$

(c) if C is not algebraically closed, roots of $m(0,y)$ may live in extension fields of C

(d) maybe not every root of $m(0,y)$ can be continued to a full series solution.

<u>Ex:</u> $m = y^2 - x$

$$m(0,y) = y^2 \Rightarrow a_0 = 0$$

$$f = 0 + a_1 x + a_2 x^2 + \cdots$$

$$f^2 = \qquad a_1^2 x^2 + \cdots$$

$$m(x,f) = -x + a_1^2 x^2 + \cdots \neq 0$$

regardless of the choice of a_1.

$\Rightarrow$ the equation $m=0$ has no solution in $\mathbb{C}[[x]]$

(the solutions $\sqrt{x}, -\sqrt{x}$ do not belong to $\mathbb{C}[[x]]$).

<u>Thm:</u> Let $m \in \mathbb{C}[x,y]$ and let $a_0 \in \mathbb{C}$ be a simple root of $m(0,y) = [x^0]m \in \mathbb{C}[y]$. Then there exists a unique fps $f \in \mathbb{C}[[x]]$ with $f(0) = a_0$ and $m(x,f) = 0$.

<u>Proof:</u> we show by induction that for all $n \in \mathbb{N}$ there exists a unique $f \in \mathbb{C}[[x]]/\langle x^n \rangle$ with $f(0) = a_0$ and $m(x,f) = 0 \mod x^n$.

$n=1$: take $f = a_0$

$n \to n+1$: ansatz $f = f_n + a_n x^n$

write $m = m_0(x) + m_1(x)y + \dots + m_r(x)y^r$

then $m(x,f) = \sum_{i=0}^{r} m_i(x) \underbrace{(f_n + a_n x^n)^i}_{= \sum_{j=0}^{i} \binom{i}{j} f_n^j \underbrace{(a_n x^n)^{i-j}}_{= 0 \bmod x^{n+1} \text{ unless } i-j \leq 1}}$

$$= \sum_{i=0}^{r} \left(f_n^i + i f_n^{i-1} a_n x^n \right) \bmod x^{n+1}$$

$$= m(x, f_n) + (D_y m)(x, f_n) a_n x^n \bmod x^{n+1}$$

So we can (and must) take

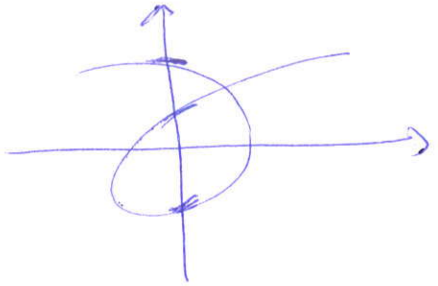
$$a_n = - \frac{[x^n]\, m(x, f_n)}{[x^n] (D_y m)(x, f_n) x^n}.$$

The denominator is nonzero because

$$[x^n](D_y m)(x, f_n) \cdot x^n$$
$$= [x^0](D_y m)(x, f_n) = (D_y m)(0, a_0) \neq 0,$$

because a_0 is a simple root. $\blacksquare$

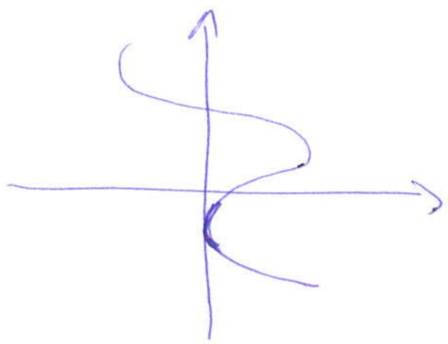
Geometrically, an equation $m = 0$ with $m \in \mathbb{C}[x,y] \setminus \{0\}$ defines a curve in $\mathbb{C}^2$



Generically, the curve intersects the y-axis in exactly $\deg_y m$ distinct points.

The thm above associates to each of these points a fps solution which we can view as Taylor expansions of a function defined in a neighborhood of 0 whose graph agrees with the curve.

Degenerate cases:

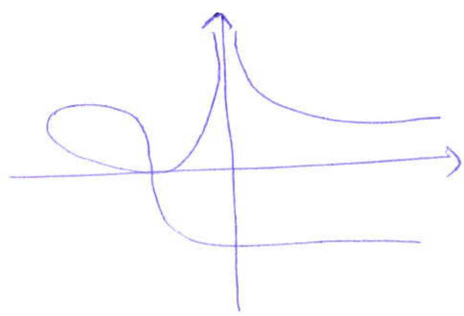


branch point

$\Leftrightarrow$ 0 is multiple root of $m(0,y)$

$\Leftrightarrow$ $\gcd(m(0,y), (D_y m)(0,y)) \neq 0$

$\Leftrightarrow$ 0 is a root of $\operatorname{res}_y(m, D_y m) = \operatorname{disc}_y(m)$



poles

$\Leftrightarrow$ $\deg m(0,y) < \deg_y m$

$\Leftrightarrow$ $(\operatorname{lc}_y m)(0) = 0$

More generally:

<u>Def:</u> Let $m \in \mathbb{C}[x,y] \setminus \{0\}$ and $\xi \in \mathbb{C}$.

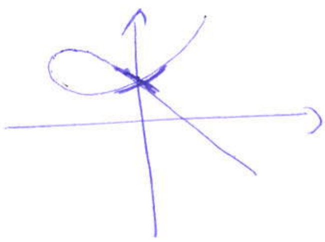
(1) $\xi \in \mathbb{C}$ is called a <u>branch point</u> for m if $m(\xi, y) \in \mathbb{C}[y]$ has multiple roots.

(2) $\xi \in \mathbb{C}$ is called a <u>pole point</u> for m iff ξ is a root of $lc_y(m)$

(3) $\xi \in \mathbb{C}$ is called a <u>singular point</u> for m iff it is a branch point or a pole point.

<u>Note:</u>

At an ordinary point there are always $\deg_y m$ many solutions in $\mathbb{C}[[x-\xi]]$. At a pole point, there are always fewer than $\deg_y m$ many fps solutions. At a branch point there can be $\deg_y m$ or fewer sols.

Ex: 

0 is a branch point where there are nevertheless two fps solutions.

Thm (Puiseux, without proof). If C is algebraically closed then for every $m \in C[x,y] \setminus \{0\}$ irreducible there exists $r \in \mathbb{N} \setminus \{0\}$ and $\deg_y m$ many formal Laurent series $f \in C((x))$ such that $m(x^r, f) = 0$.

Ex: For $m = y^2 - x$ we can take $r = 2$ because $y^2 - x^2$ has two solutions $y = \pm x \in C[[x]]$. We can also say that $y = \pm x^{1/2} \in C[[x^{1/2}]]$ are solutions of $m = y^2 - x$. Such series are called Puiseux series.

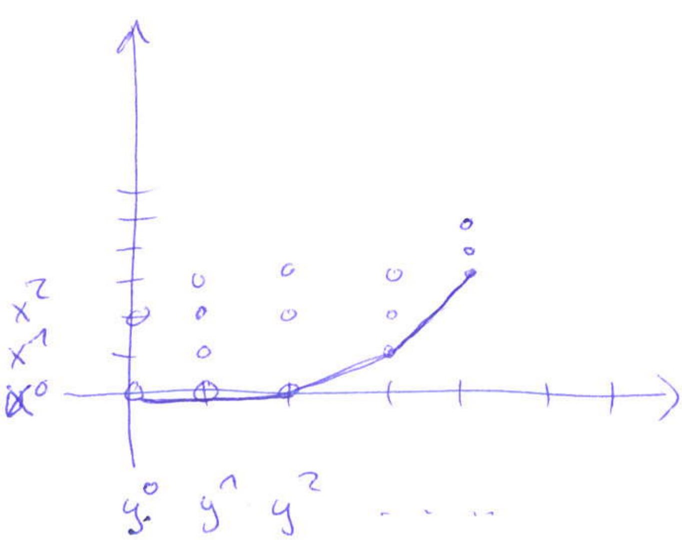
Puiseux's theorem says more generally that $\bigcup_{r \in \mathbb{N} \setminus \{0\}} \mathbb{C}((x^{1/r}))$ is an algebraically closed field.

The terms of a Puiseux series solution for a given $m \in \mathbb{C}[x, y] \setminus \{0\}$ can be computed recursively. Make an ansatz $f = f_0 + c x^\alpha + \ldots$ with unknown $c \in \mathbb{C}$ and $\alpha \in \mathbb{Q}$, suppose f_0 contains all the terms already computed (start with $f_0 = 0$). Then $m(x, f_0 + c x^\alpha)$ for symbolic c, α is a $\mathbb{C}$-linear combination of terms of the form $c^u x^{u\alpha + v}$ with $u \in \mathbb{N}, v \in \mathbb{Q}$. For any specific choice of $\alpha \in \mathbb{Q}$, one of these terms will have a minimal exponent, so in order for f to be a solution we must have $c = 0$, unless there is more than one contribution to the lowest order term.

This is the case when $\alpha \in \mathbb{Q}$ is such that $u_i\alpha + v_i = u_j\alpha + v_j \leq u_k\alpha + v_k$ for some $i \neq j$ and all k. So the eligible values of α are among the numbers $-\frac{v_i - v_j}{u_i - u_j}$ $(i \neq j)$, of which there are only finitely many.

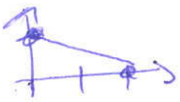
Each of these values leads to at least one monomial in $m(x, f_0 + cx^\alpha)$ whose coefficient is a nontrivial polynomial in c. Pick the α's where this is the lowest order term and α is greater than the exponents in f_0. (There will be such a choice). The roots of the coefficient polynomial are the possible choices for c.

Geometric procedure: eligible values for α are the slopes in the bottom part of the convex hull of the support of $m(x, f_0 + y)$, times -1.



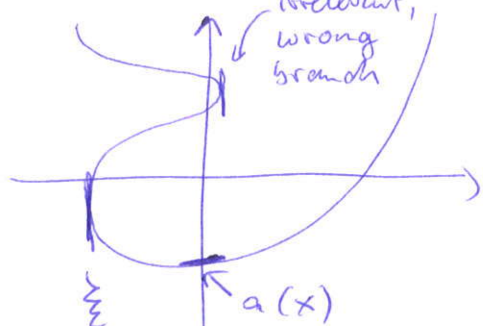
(Newton Polygon)

Ex: $m = y^2 - x$



slope $= -\frac{1}{2} \Rightarrow \alpha = +\frac{1}{2}$.

Why should we care about Puiseux series solutions? In analytic combinatorics, counting sequences $(a_n)_{n=0}^{\infty}$ are represented by fps $a(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]]$ ("generating functions") which are often algebraic. (Ex: a_n = # of binary trees with n nodes $\Rightarrow x\,a(x)^2 - a(x) + 1 = 0$) If such a fps is viewed as complex function, then the asymptotic behaviour of the coeff sequence $(a_n)_{n=0}^{\infty}$ $(n \to \infty)$ is determined by the nearest Puiseux series expansion at the nearest singularity.



Fact: If

$$a(1-\tfrac{x}{\xi}) = \underbrace{\dots}_{\text{terms with exponents in } \mathbb{N}} + c_k\left(1-\tfrac{x}{\xi}\right)^{k/r} + \underbrace{\dots}_{\text{terms with exponents} > k/r}$$

then

$$a_n \sim \frac{c_k}{\Gamma(-\frac{k}{r})}\left(\frac{1}{\xi}\right)^n n^{-1-\frac{k}{r}} \quad (n \to \infty)$$

See Flajolet/Sedgewick for details