# D-finite Functions

Def.

(1) A function $f$ is called D-finite if there are polynomials $p_0, \dots, p_r$, $p_r \neq 0$ such that

$$p_0(x) f(x) + p_1(x) f'(x) + \dots + p_r(x) f^{(r)}(x) = 0$$

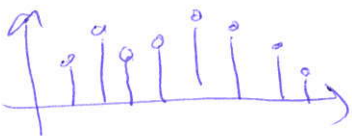
Such an equation is called a (linear) (ordinary) differential equation (of order $r$) (with polynomial coefficients)

(2) A sequence $(a_n)$ is called D-finite (or P-finite, P-recursive) if there are polynomials $p_0, \dots, p_r$, $p_r \neq 0$, such that

$$p_0(n) a_n + p_1(n) a_{n+1} + \dots + p_r(n) a_{n+r} = 0$$

Such an equation is called a (linear) (ordinary) recurrence (of order $r$) (with polynomial coefficients)

<u>Context</u>: We want to do exact computations with functions: "given a function, compute ... or decide ...". What means "given a function"? In calculus classes, it typically means given an expression for a function, e.g. $f(x) = \sqrt{\frac{1-x}{\log(x+e^{-x^2})}}$. In applications, it typically means approximate data for values of the function at certain points



<u>Problems</u>:

(1) many interesting functions cannot be expressed in closed form

(2) approximate data is not exact.

We need a good data structure for representing functions.

Fundamental problem: This would mean that we could encode each function by a finite bit string. The set of bit strings is countable, while the set of functions (say $\mathbb{C} \to \mathbb{C}$) is uncountable. Thus no good data structure exists.

In order to do exact computations with functions, we must limit ourselves to a certain class of functions. The class should not be too small (otherwise it won't be very useful) and not to big (otherwise computations become to hard). Main message of this course: The class of D-finite functions forms a good compromise.

Ex:

(1) $f(x) = x^2$ $\qquad x f'(x) - 2f(x) = 0$

(2) $f(x) = e^x$ $\qquad f'(x) - f(x) = 0$

(3) $f(x) = \sqrt{x}$ $\qquad x f'(x) - \frac{1}{2} f(x) = 0$

(4) $f(x) = \log(x)$ $\qquad x f''(x) + f'(x) = 0$

(5) $f(x) = \int e^{-x^2} dx$ $\qquad f''(x) + 2x f'(x) = 0$

(6) $f(x) = J_3(x)$ (3rd Bessel fct) $\qquad x^2 f''(x) + x f'(x) + (x^2 - 9) f(x) = 0$

⋮

Ex:

(1) $a_n = n^2$ $\qquad$ $n^2 a_{n+1} - (n+1)^2 a_n = 0$

(2) $a_n = 2^n$ $\qquad$ $a_{n+1} - 2a_n = 0$

(3) $a_n = \sum_{k=1}^{n} \frac{1}{k} =: H_n$ $\qquad$ $(n+2)a_{n+2} - (2n+3)a_{n+1} + (n+1)a_n = 0$

(harmonic number)

(4) $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{2n}{2k}$

$$\bigcirc\, a_{n+3} + \bigcirc\, a_{n+2} + \bigcirc\, a_{n+1} + \bigcirc\, a_n = 0$$

polys of degree 4.

Where do the functions and sequences live?
There are several options

(a) Analytic/meromorphic functions $U \to \mathbb{C}$ ($U \neq \emptyset$) for some open set $U \subseteq \mathbb{C}$

(b) Formal power series (fps) $C[[x]]$ for some field $C$ (characteristic 0)

(c) A differential ring

(a) Analytic/meromorphic functions $U \to \mathbb{C}$ ($U \neq \emptyset$) for some open set $U \subseteq \mathbb{C}$ with $\mathbb{Z} \subseteq U$

(b) sequences $\mathbb{N} \to C$ or $\mathbb{Z} \to C$ for some field $C$

(c) A difference ring.

# 1. Preliminaries

Throughout this course, $C$ is a field of characteristic $0$, e.g. $C = \mathbb{Q}$

Def: Let $R$ be a ring

(1) A map $D: R \to R$ is called a derivation on $R$ if

$\forall a, b \in R$: $D(a+b) = D(a) + D(b)$

$D(ab) = D(a)b + aD(b)$

(2) If $D$ is a derivation on $R$ then the pair $(R, D)$ is called a differential ring (differential field if $R$ is a field)

(3) If $(R, D)$ is ~~called~~ a differential ring then

$$\mathrm{Const}(R) := \{ c \in R \mid D(c) = 0 \}$$

is called the set of constants of $R$.

Ex:

(1) The set of all analytic functions $U \to \mathbb{C}$ together with pointwise + and · forms a ring which together with the usual derivation $\frac{d}{dz}$ becomes a differential ring. If $U$ is connected, its constants are the constant functions.

(2) $C[x]$ together with the usual + and · and $D$ is a differential ring. So is $C[x]$.

(3) $C(x,y)$ together with the unique derivation defined by $D(c)=0\ \forall c \in C$, $D(x)=1$, $D(y)=y$ is a differential field. (Observe: $y$ behaves like $e^x$)

Facts:

(1) $\forall a \in R\ \forall n \in \mathbb{N}$: $D(a^n) = n a^{n-1} D(a)$.
Also works for all $n \in \mathbb{Z}$ when $a \in R^\times$.

(2) $\mathrm{Const}(R)$ is a subring of $R$ (and a subfield if $R$ is a field)

Def: Let $R$ be a ring

(1) A map $\sigma: R \to R$ is called an endomorphism if

$\forall a,b \in R$: $\sigma(a+b) = \sigma(a) + \sigma(b)$
$\sigma(ab) = \sigma(a)\sigma(b)$

(2) If $\sigma$ is an endomorphism on $R$ then the pair $(R,\sigma)$ is called a difference ring. (difference field if $R$ is a field)

(3) If $(R,\sigma)$ is a difference ring, then $\mathrm{Const}(R) := \{ c \in R \mid \sigma(c) = c \}$ is called the set of constants of $R$.

Ex:

(1) The ring $C^{\mathbb{N}}$ of sequences $\mathbb{N} \to C$ together with pointwise $+$ and $\cdot$ and the shift operator $\sigma$ defined by

$\sigma((a_n)_{n=0}^{\infty}) := (a_{n+1})_{n=0}^{\infty}$

is a difference ring. Its constants are the constant sequences.

(2) The set $I = \{(a_n) \mid \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 : a_n = 0\}$ is an ideal of $C^{\mathbb{N}}$ (i.e. $\forall a, b \in I : a+b \in I$, $\forall a \in I\, \forall b \in R : ab \in I$). It is also closed under $\sigma$ (i.e. $\forall a \in I : \sigma(a) \in I$). Therefore

$$\bar{\sigma} : R/I \to R/I ; \quad \bar{\sigma}([(a_n)]_\sim) := [\sigma((a_n))]_\sim$$

is well defined, and $(R/I, \bar{\sigma})$ is also a difference ring. Its elements are called <u>germs</u> of sequences (at infinity)

(3) $C(x,y)$ together with the unique endomorphism defined by $\sigma(c) = c$ $\forall c \in C$, $\sigma(x) = x+1$, $\sigma(y) = 2y$ is a difference field (observe that $y$ behaves like $2^x$).

<u>Def:</u> A ring (field) $R$ is called <u>computable</u> iff

(1) every element of $R$ admits a finite representation (not necessarily unique)

(2) Representations for $0$ and $1$ are known

(3) There are algorithms which for given representations of $a, b \in R$ compute a representation of $a+b$, $a-b$ and $a \cdot b$ (and, in the case of a field, $a/b$ if $b \neq 0$)

(4) There is an algorithm which decides for a given representation of $a \in R$ whether $a \stackrel{?}{=} 0$

<u>Ex:</u>

(1) $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Z}$, $\mathbb{Z}_p$ are computable

(2) $\mathbb{R}$, $\mathbb{C}$ are not computable

(3) If $C$ is computable then so is $C(x)$.

(4) $C[\![x]\!]$ is not computable, even if $C$ is.

In order to compute in $\mathbb{C}[\![x]\!]$, there are two main approaches:

(a) <u>lazy power series</u>: restrict to the class of all fps $a \in C[\![x]\!]$ such that there is an algorithm that for any given $N \in \mathbb{N}$ computes the coefficient $[x^N]a$ of $x^N$ in $a$. This class is closed under $+, \cdot, D$, but it is not computable, because zero equivalence is not decidable.

(b) <u>truncated series</u>: fix an $N \in \mathbb{N}$ (called "precision") and compute in $C[\![x]\!]/\langle x^N \rangle$ instead of $C[\![x]\!]$. Here $\langle x^N \rangle$ is the ideal consisting of all fps of the form

$$0 + 0x + \cdots + 0x^{N-1} + \circledast x^N + \circledast x^{N+1} + \cdots$$

Some more facts on formal power series:

(1) $a \in \mathbb{C}[\![x]\!]$ admits a multiplicative inverse $\frac{1}{a}$ in $\mathbb{C}[\![x]\!] \Leftrightarrow [x^0]a \neq 0$

(2) for $a, b \in \mathbb{C}[\![x]\!]$ with $[x^0]b = 0$ there exists a composition $a \circ b$ in $\mathbb{C}[\![x]\!]$

(3) for $a \in \mathbb{C}[\![x]\!]$ with $[x^0]a = 0$ and $[x^1]a \neq 0$ there exists a unique $b \in \mathbb{C}[\![x]\!]$ with $a \circ b = x$

All these operations can be executed with the truncated series as well as the lazy series paradigm.