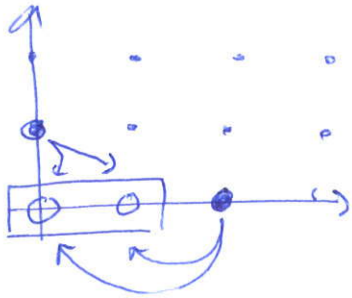
$f = e^x + y$ is D-finite, because

$$D_x^2 f = ☁\, D_x f + ☁\, f$$
$$D_y f = ☁\, D_x f + ☁\, f$$



Recall:

(1) An Ore algebra $\mathbb{O} = K[\partial_1, \dots, \partial_n]$ acts on a function space $\mathcal{F}$

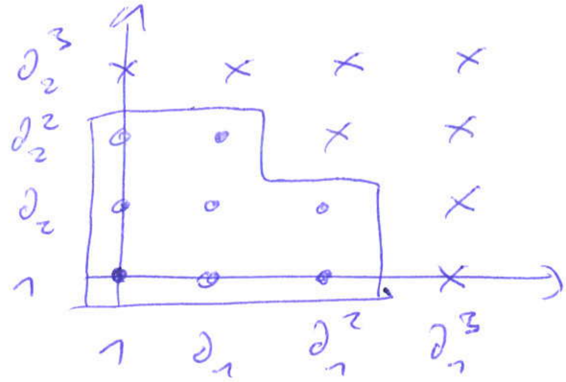
eg $K = \mathbb{C}(x,y)$, $\partial_1 = \frac{d}{dx}$, $\partial_2 = \frac{d}{dy}$,
$\mathcal{F}$ = bivariate meromorphic functions

(2) $f \in \mathcal{F}$ is D-finite $\Leftrightarrow$

$$\dim_K \underbrace{\mathbb{O}/\mathrm{ann}(f)}_{\simeq\, \mathbb{O}\cdot f \,\subseteq\, \mathcal{F}} < \infty$$

(3) f is D-finite $\Leftrightarrow$ there is a choice of finitely many monomials $\partial_1^{e_1} \cdots \partial_n^{e_n}$ (the bullets below) so that

every $L \in \mathbb{O}$ is equivalent mod $\operatorname{ann}(f)$ to a linear combination of these.



(4) If there are no issues with singularities, a D-finite fps is uniquely determined by $\operatorname{ann}(f)$ and the (finitely many!) coeffs of the terms $x_1^{e_1} \dots x_n^{e_n}$ where $(e_1 \dots e_n)$ is a bullet.

<u>Thm:</u> $f \in \mathcal{F}$ is D-finite

$\iff \forall i \in \{1..n\}: \operatorname{ann}(f) \cap K[\partial_i] \neq \{0\}$

<u>Proof:</u>

"$\Rightarrow$" Let $i \in \{1..n\}$ and consider

$[1], [\partial_i], [\partial_i^2], \dots \in \mathbb{O}/\operatorname{ann}(f)$.

Since $\dim_K \mathbb{O}/\operatorname{ann}(f) < \infty$, there is an $r \in \mathbb{N}$ such that $[1], \dots, [\partial_i^r]$ are linearly dependent over K, say $p_0[1] + \dots + p_r[\partial_i^r] = 0$ for some $p_0, \dots, p_r \in K$, not all zero. Then $\underbrace{p_0 + p_1\partial_i + \dots + p_r\partial_i^r}_{\in K[\partial_i] \setminus \{0\}} \in \mathrm{ann}(f)$, as required.

"$\Leftarrow$" If r_i is the order of the generator of $\mathrm{ann}(f) \cap K[\partial_i]$, then $\dim_K \mathbb{O}/\mathrm{ann}(f) < r_1 \cdots r_n$.

By this thm, it is always possible to represent a D-finite function by "prime" operators, one for each ∂_i. But this is usually not a good choice. It is much more economic to use instead a Gröbner basis of $\mathrm{ann}(f)$.

Def:

(1) A total order $\leq$ on $\mathbb{N}^n$ is called admissible if $(0,\dots,0)$ is the smallest element and $u \leq v \Rightarrow u+s \leq v+s \quad \forall u,v,s \in \mathbb{N}^n$.

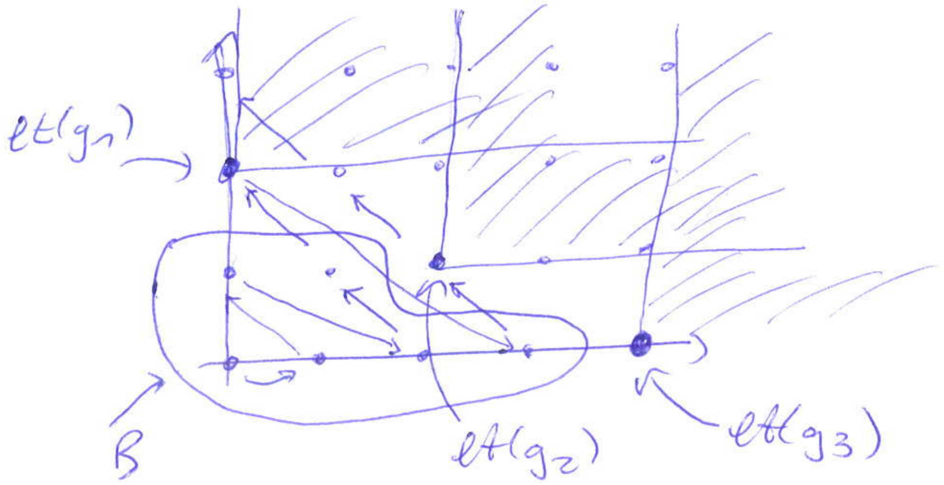
(2) Let $\leq$ be an admissible order on $\mathbb{N}^n$ and $L \in K[\partial_1,\dots,\partial_n] \setminus \{0\}$. The term $\partial_1^{e_1}\cdots\partial_n^{e_n}$ of L whose exponent vector $(e_1 \dots e_n)$ is maximal wrt $\leq$ is called the leading term of L, denoted by $lt(L)$.

(3) Let $I \subseteq K[\partial_1,\dots,\partial_n]$ and let $G \subseteq I$ be a basis of I (i.e. a set of generators, not nec independent in any way). Let

$$B := \{ [\partial_1^{e_1}\cdots\partial_n^{e_n}] \mid \forall g \in G : lt(g) \nmid \partial_1^{e_1}\cdots\partial_n^{e_n} \}.$$

G is called a Gröbner basis (of I) if B is a K-VS basis of $\mathcal{O}/I$.

Ex:



There is an algorithm which transforms any given finite ideal bases of I into a Gröbner basis (Buchberger's algorithm). In the present context, this is not as important as it is in commutative algebra and algebraic geometry, because we can often ensure that we obtain a Gröbner basis right away.

Thm: Let $f, g \in \mathcal{F}$ be D-finite.

(1) $f + g$ is D-finite

(2) If $\mathcal{F}$ is a ring, also $f \cdot g$ is D-finite.

For (1), observe that

$$\operatorname{ann}(f) \cap \operatorname{ann}(g) \subseteq \operatorname{ann}(f+g)$$

because every operator which kills both f and g also kills any $\mathbb{C}$-linear combination of them.

Here is a way to compute a Gröbner bases of $\operatorname{ann}(f) \cap \operatorname{ann}(g)$ given Gröbner bases of $\operatorname{ann}(f)$ and $\operatorname{ann}(g)$. This approach is known as FGLM.

(1) $B = \emptyset$, $G = \emptyset$

(2) while there exist terms which are not in B and not a multiple of a leading term of an element of G, do:

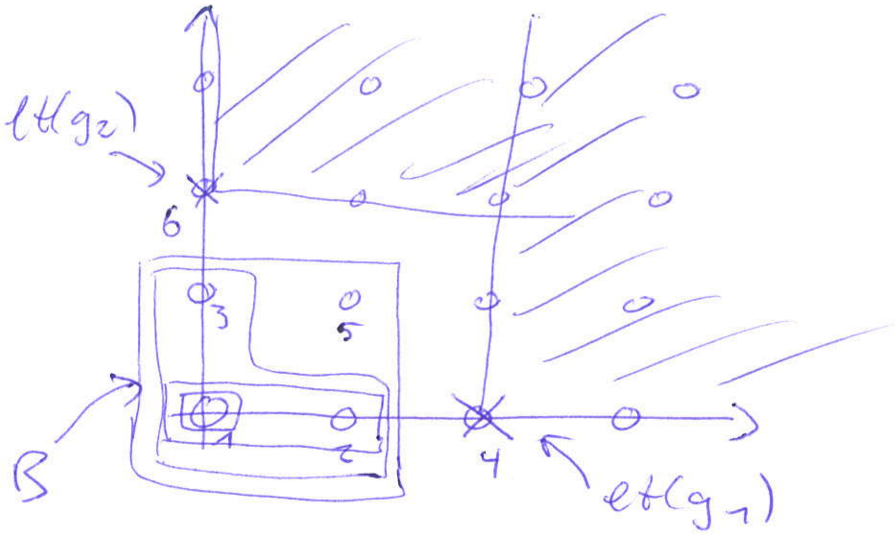
(3) let τ be the smallest such term (wrt the given order)

(4) find an operator with support in $B \cup \{\tau\}$ which belongs to $\operatorname{ann}(f) \cap \operatorname{ann}(g)$

(5) If there is one, add H to G, otherwise, add τ to B.

(6) return G.

Ex:



Step 4 can be realized by linear algebra. Make an ansatz

$$L = \tau + \sum_{b \in B} p_b b$$

with undetermined $p_b \in K$, rewrite H as K-linear combinations of both $\mathcal{O}/\mathrm{an}(f)$ and $\mathcal{O}/\mathrm{an}(g)$ and force the coeffs to be equal to zero. This gives a linear system with $|B|$ variables and $\dim_K \mathcal{O}/\mathrm{an}(f) + \dim_K \mathcal{O}/\mathrm{an}(g)$ equations, which we can solve.

It is not hard to see that B remains K-linearly independent throughout the algorithm, which implies that the output is indeed a Gröbner basis. Furthermore, since $\dim_K \mathbb{O}/\mathrm{ann}(f) + \dim_K \mathbb{O}/\mathrm{ann}(g)$ ~~are~~ is finite, the algorithm must terminate, because ~~the~~ $|B|$ cannot exceed this bound.

Algorithms for other closure properties work similarly.

10 Creative Telescoping

Recall:

If $f(x)$ is D-finite, so is $\int f(x)\,dx$

If a_n is D-finite, so is $\sum_{k=0}^{n} a_k$.

Similarly, it is also true that

If $f(x,y)$ is D-finite, so is $F(x,y) = \int_0^x f(t,y)\,dt$

If $a_{n,k}$ D-finite, then $b_{n,k} = \sum_{i=0}^{n} a_{i,k}$ too.

It is a somewhat different question to ask whether also

$$F(y) = \int_0^1 f(x,y)\,dx \quad \text{and} \quad b_n = \sum_{k=0}^{n} a_{n,k}$$

are D-finite.

In the former case, we speak about "indefinite" summation / integration (easy), in the latter case, we call it "definite" summation / integration (harder).

<u>Ex:</u> $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ D-finite?

<u>Idea:</u> In order to construct an operator which kills $\sum_k a_{n,k}$, when we know ann $(a_{n,k})$, we would look for an element

$$L = \underset{\substack{\cap \\ C(n)[S_n]\setminus\{0\} \\ \text{"telescoper"}}}{P} + (S_k - 1)\underset{\substack{\cap \\ C(n,k)[S_n, S_k] \\ \text{"certificate"}}}{Q} \in \operatorname{ann}(a_{n,k})$$

If we find one, then

$$P \cdot a_{n,k} + (S_k - 1)\, Q \cdot a_{n,k} = 0$$

$$\Rightarrow \underbrace{\sum_{k=0}^{n} P \cdot a_{n,k}}_{= P \cdot \sum_{k=0}^{n} a_{n,k}} + \underbrace{\sum_{k=0}^{n} (S_k - 1)\, Q \cdot a_{n,k}}_{= \underbrace{[Q \cdot a_{n,k}]_{k=0}^{n+1}}_{\text{D-finite, usually } 0}} = 0$$

because P is free of k, S_k

$$\Rightarrow \sum_{k=0}^{n} a_{n,k} \text{ is D-finite.}$$

The reasoning in the differential case is similar, using $\int_0^1 D_x\, Qf\, dx = [Qf]_{x=0}^{1}$

Warning: You need to make sure that $Q \cdot f$ has no poles in the integration/summation range. This is not ensured automatically.

How to find P and Q when $\mathrm{ann}(f)$ is known? There are four generations of algorithms for doing so:

(1) (since ~1940) find L by elimination, i.e. compute $\underbrace{\mathrm{ann}(f)}_{\in C[x,y][D_x, D_y]} \cap C[x][D_x, D_y]$

(2) (since ~1990) Zeilberger-type algorithms: make an ansatz for P and pipe it through a summation/integration algorithm for finding the corresponding Q. Keep repeating with larger and larger P until a Q is found.

(3) (since ~ 2005) Apagodu-Zeilberger-type algorithms: make an ansatz for both P, Q and solve a linear system.

(4) (since ~ 2010) reduction-based approach: compute certain "normal forms" of $1, \partial_x, \partial_x^2, \ldots$ and find linear dependencies between them.

<u>ad3</u> Assume for simplicity that the integrand is a rational function $f = \frac{u}{v} \in C(x,y)$ rather than an arbitrary D-finite function. We seek $P \in C(x)[D_x] \setminus \{0\}$ and $Q \in C(x,y)[D_x, D_y]$ such that

$$(P - D_y Q) \cdot f = 0$$

Writing $g = Q \cdot f$, we may as well search for g instead of Q and then take $Q = \frac{g}{f} \in C(x,y)[D_x, D_y]$.

Ansatz for P: $p_0 + p_1 D_x + \cdots + p_r D_x^r$.

$$f = \frac{u}{v}$$

$$D_x f = \frac{u'v - uv'}{v^2}$$

$$D_x^2 f = \frac{\bigcirc' v - 2\bigcirc v'}{v^3}$$

$$\vdots$$

$$D_x^r f = \frac{\bigcirc}{v^{r+1}} \leftarrow \deg_y \le \deg_y u + r \deg_y v$$

so

$$(p_0 + p_1 D_x + \cdots + p_r D_x^r) \cdot f = \frac{\bigcirc}{v^{r+1}}$$

$$\overset{!}{=} D_y \underbrace{\frac{q_0 + q_1 y + \cdots + q_d y^d}{v^r}}_{\text{ansatz for } g}$$

Let's take $d = \deg_y u + (r-1)\deg_y v$, to make the numerator degrees match. Then comparing coefficients gives a linear system with

$$\#\text{vars} = \underbrace{(r+1)}_{\text{from } P} + \underbrace{(d+1)}_{\text{from } g}$$

$$\#\text{equs} = d + 1 + \deg_y v$$

So there will be a solution as soon as $\boxed{r+1 > \deg_y v}$

Minor bug: "nontrivial" solution means that the pair (P, g) is nonzero. We want that P is nonzero. Could there be a nontrivial solution with $P=0$? For such a solution, we would have $D_y g = 0$. This can indeed happen, namely when $g \in C$. But then the numerator of $g = \frac{\quad}{v^r}$ must be a constant multiple of v^r. We can exclude this solution by removing the term $q_i y^i$ with $i = r \deg_y v$ from the ansatz. This costs one variable but otherwise does not affect the analysis. So there always exists a P of order $\boxed{r > \deg_y v}$.

ad 4 Recall: every ratfun $f \in C(y)$ can be written as

$$f = D_y(g) + h$$

for some $g, h \in C(y)$ with

$$\operatorname{denom}(h) = \operatorname{sqfp}(\operatorname{denom}(f)).$$

$$\deg_y(\operatorname{num}(h)) < \deg_y(\operatorname{denom}(h)).$$

($\to$ "Hermite reduction")

Now, for $f = \frac{u}{v} \in C(x,y)$ we have

$$\begin{aligned} f &= D_y(g_0) + h_0 \\ D_x f &= D_y(g_1) + h_1 \\ &\vdots \\ D_x^r f &= D_y(g_r) + h_r \end{aligned}$$

$$\sum_{i=0}^{r} p_i D_x^i f = D_y(\bigcirc) + \sum_{i=0}^{r} p_i h_i.$$

It thus suffices to find $p_0, \dots, p_r \in C(x)$, not all zero, with $\sum_{i=0}^{r} p_i h_i = 0$.

Observe that all h_i have sqfp(denom(f)) as denominator. Clearing this denominator and comparing coeffs wrt powers of y gives a linear system over $C(x)$ with $r+1$ vars and $\deg_y$ sqfp(denom(f)) variables, so there is a solution as soon as

$$\boxed{r + 1 > \deg_y \operatorname{sqfp}(\operatorname{denom}(f))}$$

Advantage of (4) over (3): We can compute P without also computing Q. When Q is not needed, this is good, because Q is much bigger than P.