8 Several Functions

Any operator equation of order r,

$$(p_0 + p_1 \partial + \dots + p_r \partial^r) \cdot f = 0,$$

can be translated into a matrix equation of order 1,

$$\partial \begin{pmatrix} f_0 \\ \vdots \\ \vdots \\ f_{r-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{p_0}{p_r} & \cdots & \cdots & \cdots & -\frac{p_{r-1}}{p_r} \end{pmatrix}}_{\text{"companion matrix"}} \cdot \begin{pmatrix} f_0 \\ \vdots \\ \vdots \\ f_{r-1} \end{pmatrix}$$

We have that f is a solution of the scalar equation $\iff (f, \partial f, \dots, \partial^{r-1} f)$ is a solution of the matrix equation.

<u>Conversely:</u> If $K[\partial]$ is an Ore algebra over a field K acting on $\mathcal{F}$, and $A \in K^{r \times r}$ is such that

$$\partial \begin{pmatrix} f_0 \\ \vdots \\ f_{r-1} \end{pmatrix} = A \begin{pmatrix} f_0 \\ \vdots \\ f_{r-1} \end{pmatrix}$$

then each coordinate function f_i is D-finite.

To see why, consider a solution vector f.

Then

$$\partial f = A f$$

$$\partial^2 f = \partial (A f) = \sigma(A)\, \partial f + \delta(A) f$$

$$= \underbrace{(\sigma(A) A + \delta(A))}_{=: A_2} \cdot f$$

$$\partial^3 f = \partial(A_2 f) = \underbrace{(\sigma(A_2) A_2 + \delta(A_2))}_{=: A_3} f$$

$$\vdots$$

The matrices $A, A_2, A_3, \dots$ all belong to $K^{r \times r}$, a K-VS of dimension r^2. Therefore, there are ~~polynomials~~ elements $p_0, \dots, p_{r^2} \in K$, not all zero, such that

$$p_0 I + p_1 A + p_2 A_2 + \dots + p_{r^2} A_{r^2} = 0$$

But then also

$$(p_0 + p_1 \partial + \dots + p_{r^2} \partial^{r^2}) \cdot f = 0$$

(coordinatewise). So all coordinate functions are D-finite.

The process of computing an annihilating operator from a given A is called uncoupling.

Ex: Suppose that $A \in K^{r\times r}$ has in fact constant entries, i.e. $A \in C^{r\times r}$. Then

$$\partial f = Af \Rightarrow \partial^2 f = \partial(Af) = A\,\partial f$$
$$= A^2 f$$

$$\Rightarrow \dots$$
$$\Rightarrow \partial^r f = A^r f.$$

By Cayley-Hamilton we have $\chi(A)=0$ for the characteristic polynomial χ of A. This implies $\chi(\partial)\cdot f = 0$.

It can be shown that also for $A \in K^{r\times r}$ (not necessarily constant) there always is an operator of order $\leq r$ which annihilates all coordinates of any solution vector $f \in \mathcal{F}^r$ of $\partial f = Af$

We don't have the time to go into further details.

9 Several Variables

<u>Goal:</u> Extend the notion of D-finiteness to functions in several variables.

<u>Ex:</u> $f(x,y) = e^x + y$ (differential case)

$f(n,k) = \binom{n}{k}$ (shift case)

$f(n,x) = x^n$ (mixed case)

etc.

<u>Recall:</u> In the univariate case, the key idea behind a D-finite object is that it is uniquely determined by an equation and a finite amount of initial values.

Def: Let R be a ring

Let $\sigma_1, \dots, \sigma_r : R \to R$ be endomorphisms.

Let $\delta_1, \dots, \delta_r : R \to R$ be such that δ_i is a σ_i-derivation (i.e. $\forall a, b \in R$:

$\delta_i(a+b) = \delta_i(a) + \delta_i(b)$, $\delta_i(ab) = \delta_i(a)b + \sigma_i(a)\delta_i(b)$)

Suppose that $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ and $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ for all i, j.

Let $R[\partial_1, \dots, \partial_r]$ be the set of polynomials in the variables $\partial_1, \dots, \partial_r$ with coeffs in R.

This set together with the usual addition and the unique (possibly noncommutative) multiplication satisfying

$$\partial_i \partial_j = \partial_j \partial_i \quad \text{and} \quad \partial_i a = \sigma_i(a)\partial_i + \delta_i(a)$$

for all i, j and all $a \in R$ is called an Ore algebra.

Ex:

(1) differential operators:

$C(x,y)[\partial_x, \partial_y]$ with

$$\partial_x x = x\partial_x + 1 \qquad \partial_y x = x\partial_y$$
$$\partial_x y = y\partial_x \qquad \partial_y y = y\partial_y + 1$$

(2) recurrence operators:

$C(x,y)[\partial_x, \partial_y]$ with

$$\partial_x x = (x+1)\partial_x \qquad \partial_y x = x\partial_y$$
$$\partial_x y = y\partial_x \qquad \partial_y y = (y+1)\partial_y$$

Convention: instead of ∂_x we will use the symbol D_x in the differential case and the symbol S_x in the shift case.

Def: Let K be a field and $K[\partial_1 \ldots \partial_n]$ be an Ore algebra over K. Let $\mathcal{F}$ be a $K[\partial_1 \ldots \partial_n]$-module. An element $f \in \mathcal{F}$ is called D-finite iff

$$K[\partial_1, \ldots, \partial_n]/\mathrm{ann}(f) \qquad (\simeq K[\partial_1 \ldots \partial_n]\cdot f \subseteq \mathcal{F})$$

is a finite dimensional K-VS.

Ex:

(1) The space $\mathcal{F} = \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ of bivariate seqs is a $\mathbb{C}[x,y][S_x, S_y]$-module, but $\mathbb{C}[x,y]$ is not a field ☹. On the other hand, $\mathcal{F}$ is not a $\mathbb{C}(x,y)[S_x, S_y]$-module, which is what we would prefer.

Technical fix: Define

$$a \sim b \iff \exists\, p \in \mathbb{C}[x,y] \setminus \{0\}\ \forall n,k \in \mathbb{N}^2 : p(n,k) = 0 \ \vee\ a(n,k) = b(n,k)$$

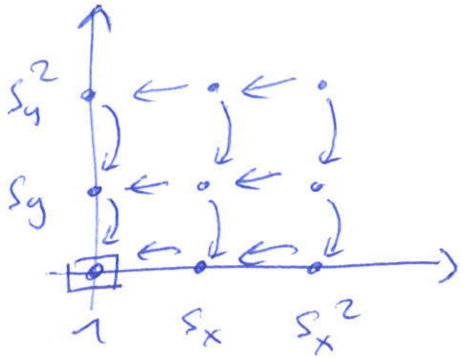
Then $\mathcal{F}/\sim$ can be viewed as $\mathbb{C}(x,y)[S_x, S_y]$-module. The equivalence class covers up possible issues emerging from vanishing denominators.

Viewed as element of $\mathcal{F}/\sim$, the binomial coefficient $f = \binom{x}{y}$ is D-finite because of the recurrences

$$\binom{x+1}{y} = \underbrace{\qquad}_{\text{rat}_1} \binom{x}{y}, \qquad \binom{x}{y+1} = \underbrace{\qquad}_{\text{rat}_2} \binom{x}{y}$$

which imply

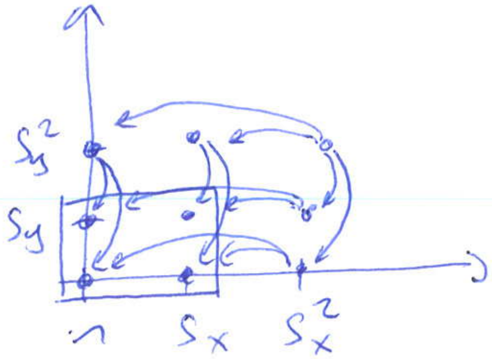
$$\dim_{C(x,y)} C(x,y)[S_x, S_y]/\operatorname{ann} f \le 1$$



Also $f = \binom{x}{y} + x^2$ is D-finite. It satisfies recurrences of the form

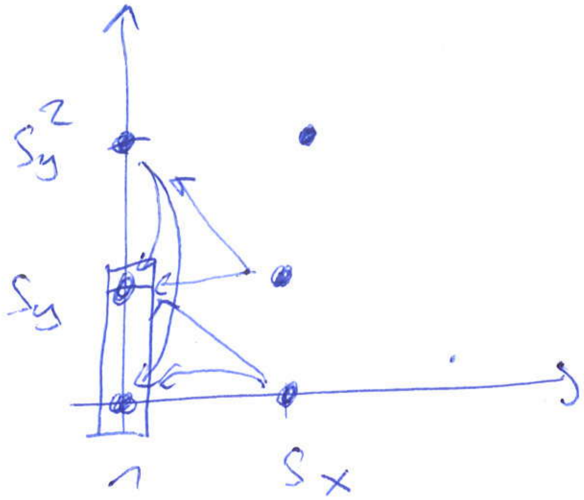
$$f(x, y+2) = \Box\, f(x, y+1) + \Box\, f(x,y)$$
$$f(x+2, y) = \Box\, f(x+1, y) + \Box\, f(x,y)$$

which imply $\dim_{C(x,y)} C(x,y)[S_x, S_y]/\operatorname{ann} f \le 4$



In fact, the dimension is only 2, because there is also a rec of the form

$$f(x+1,y) = \quad f(x,y+1) + \quad f(x,y)$$



<u>Note:</u>

(1) When there are no issues with singularities, then a D-finite seq in $C^{\mathbb{N}^n}$ is uniquely determined by $\dim_K \mathcal{O}/\mathrm{ann}$ many initial values (those corresponding to the points in the box; here $\mathcal{O}$ refers to the Ore algebra).

(2) In practice we often do not know $\mathrm{ann}(f)$ exactly but only some left ideal I with $I \subseteq \mathrm{ann}(f)$

and $\dim_K \mathcal{O}/I < \infty$. For most purposes, this is good enough.

(3) not all recurrences go well together, eg

$$f(x+1,y) = x\, f(x,y), \quad f(x,y+1) = x\, f(x,y)$$

implies

$$f(x+1,y+1) = \begin{cases} (x+1)\, f(x+1,y) = x(x+1)\, f(x,y) \\ x\, f(x,y+1) = x^2\, f(x,y) \end{cases} \lightning$$

A system of eqs/recs is called inconsistent if the ideal generated by the corresponding operators contains 1. Such a system can only have the trivial solution.

<u>Ex</u> (2) $\mathcal{F} = \mathbb{C}[[x,y]]$ is also not a good choice if we insist in a ground field $\mathbb{C}(x,y)$. But $\mathcal{F} = \mathbb{C}((x,y))$ (bivariate formal Laurent series) works. Then

$f = e^x + y$ is D-finite, because

$$D_x^2 f = \quad D_x f + \quad f$$

$$D_y f = \quad D_x f + \quad f$$