Experimental Design

Unit 9

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13.1 Introduction to two-level factorial experiments

Even a moderate number of factors can generate more treatments than can practically be included in many experiments.

The total experiment size n of the two-level design comprised of complete replicates, whether blocked or not, must be a multiple of 2^f.

Fractional factorial designs can be much smaller because they do not include all possible treatments.

To supply the information associated with omitted treatments, additional assumptions need to be made about the relationships between the treatments of interest.

These assumptions are based on the idea that most of the differences between factorial treatments can often be described in terms of main effects and interactions of relatively low order.

Fractional factorials provide partial but valuable information about treatment comparisons if further assumptions cannot be made.

13.2 Regular fractional factorial designs

A two-level *regular fractional factorial design* consists of the treatments *in one block* of a regular blocked full 2^f experiment.

For example, a „one quarter fraction of a complete 2^5 factorial experiment“ can be formed by constructing a 2^{5-2} blocked design of the 32 treatments in four blocks of size 8, and then using only one of those 4 blocks as the entire (unblocked) experimental design.

That means the levels of the factorial effects confounded with the block and the levels of their generalized interaction are fixed for the whole experiment and we cannot estimate the confounded effects.

Even more information is lost because fewer treatments are evaluated. Specifically, the 2^5 treatment structure implies that there are $2^5 - 1$ factorial effects of interest, or $2^5 - 4 = 28$ factorial effects after „sacrificing“ the confounded effects. But the fractional factorial plan contains only eight treatments, i.e. we can just estimate 8 of these factorial effects.

Example: 2^4 design with blocks of size 2^2

treatment	A	B	C	D	AB (confounded with blocks)	ACD (confounded with blocks)	BCD (confounded with blocks)	blocks
(1)	−	−	−	−	+	−	−	1
cd	−	−	+	+	+	−	−	1
abc	+	+	+	−	+	−	−	1
abd	+	+	−	+	+	−	−	1
a	+	−	−	−	−	+	−	2
bc	−	+	+	−	−	+	−	2
bd	−	+	−	+	−	+	−	2
acd	+	−	+	+	−	+	−	2
b	−	+	−	−	−	−	+	3
ac	+	−	+	−	−	−	+	3
ad	+	−	−	+	−	−	+	3
bcd	−	+	+	+	−	−	+	3
c	−	−	+	−	+	+	+	4
d	−	−	−	+	+	+	+	4
ab	+	+	−	−	+	+	+	4
abcd	+	+	+	+	+	+	+	4

ACD and BCD were used to define the blocks, these effects are confounded with blocks as is their generalized interaction AB.

Example: 2^4 design with blocks of size 2^2 (cont.)

We could now choose any of the 4 blocks for the regular fractional factorial design but should take care of aliasing.

Effects are aliased within a block, if they have the same pattern of $+$ and $-$ within the block.

E.g. in block 1 effect A is aliased with effect B, that is, the contrast in cell means associated with α is exactly the same as that associated with β. In this case we would not be able to separate the influence of these two effects using data from the fractional factorial experiment defined by block 1.

A is obviously aliased with B in blocks 1 and 4. But A is also is negatively aliased with B in blocks 2 and 3.

In a regular fractional factorial plan all factorial effects are aliased in groups. We may use the „*generator*“ of the block to identify the aliased groups.

The generator is specified by combining the confounded effects (and their generalized interaction) with the constant sign they have in the block

Example: 2^4 design with blocks of size 2^2 (cont.)

block	generator
1	I=+AB=−ACD=−BCD
2	I=−AB=+ACD=−BCD
3	I=−AB=−ACD=+BCD
4	I=+AB=+ACD=+BCD

We are interested in the effects aliased with the main effects, so we „multiply“ the main effects with the generator of the chosen block.

We choose block 2 with generator $\mathrm{I} = -\mathrm{AB} = +\mathrm{ACD} = -\mathrm{BCD}$ for the regular fractional factorial plan:

effect	product	aliased effects
A	AI=−AAB=+AACD=−ABCD	A,−B,+CD,−ABCD
C	CI=−ABC=+ACCD=−BCCD	C,−ABC,+AD,−BD
D	DI=−ABD=+ACDD=−BCDD	D,−ABD,+AC,−BC

Example: 2^4 design with blocks of size 2^2 (cont.)

For block 2 the alias groups of effects are

treat ment	aliased group 1				aliased group 2				aliased group 3				confounded with blocks		
	A	B	CD	ABCD	C	AD	BD	ABC	D	AC	BC	ABD	AB	ACD	BCD
a	+	−	+	−	−	−	+	+	−	−	+	+	−	+	−
bc	−	+	−	+	+	+	−	−	−	−	+	+	−	+	−
bd	−	+	−	+	−	−	+	+	+	+	−	−	−	+	−
acd	+	−	+	−	+	+	−	−	+	+	−	−	−	+	−

These alias groups form a partition of all the factorial effects.

The 3 degrees of freedom available for among-treatment analysis represent the information that can be gained concerning the 3 alias groups after removing the group of the effects confounded with the block.

13.3 Analysis

The result of the 2^{4-2} fractional factorial experiment is 4 estimable *strings* of effects, i.e., the linear combination of each group of aliased effects in which the coefficients are +1 and -1 to represent positive and negative aliasing.

The fourth alias group is the group of the effects confounded with the block, which includes I (or μ). For example, we noted above that one alias group is

$$\mathrm{A} \;=\; -\mathrm{B} \;=\; +\mathrm{CD} \;=\; -\mathrm{ABCD}$$

As a result, the data contrast that is the usual estimate of α is really an estimate of a „string“ of effects:

$$\mathsf{E}(\hat{\alpha}) = \alpha - \beta + (\gamma\delta) - (\alpha\beta\gamma\delta)$$

The individual effects that are most likely „real“ must be identified by other information - expert knowledge, hierarchy or heredity rules, and/or further experiments.

13.5 Comparison of fractions
13.5.1 Resolution

Some fractional factorial designs are better than others. *Resolution* is an index used to compare fractional factorial designs for overall quality of the inferences that can be drawn, and is defined as the length of the shortest word (or order of the lowest-order effect) aliased with „I“ in the generating relation.

The resolution of a design is sometimes denoted by a subscripted roman numeral.

E.g. the fractional factorial design above formed as the 2^{nd} block of the complete 4-factor design denoted by $\mathrm{I} = -\mathrm{AB} = +\mathrm{ACD} = -\mathrm{BCD}$ would be denoted as a 2_{II}^{4-2} fractional factorial plan.

Fractions with large resolution are generally desirable.

Resolution (cont.)

Consider some general cases:

- resolution II: a 2^{nd} order interaction is in the generator e.g. +AB. Then $\mathrm{A} = +\mathrm{B}$, i.e. at least some pairs of main effects are aliased.
- resolution III: main effects will not be aliased with each other, but at least some will be aliased with two-factor interactions.
- resolution IV: main effects will not be aliased with each other or with two-factor interactions, these plans support unbiased estimation of all main effects even if two-factor interactions are present.
- resolution V: these plans support estimation of a complete second-order model, and supply unbiased estimates when that model is correct.

Generally in a design of resolution R, no O-order effect is confounded with any effect of order less than R-O.

Resolution (cont.)

The fractional factorial designs most often used in practice are of resolution III, IV, and V, since designs of resolution II cannot separate the influence of all main effects, and designs of resolution VI and larger often require more than a practical number of units.

Number of Factors

Runs	2	3	4	5	6	7	8	9	10	11
4	2^{2}	2^{3-1}_{III}								
8		2^{3}	2^{4-1}_{IV}	2^{5-2}_{III}	2^{6-3}_{III}	2^{7-4}_{III}				
16			2^{4}	2^{5-1}_{V}	2^{6-2}_{IV}	2^{7-3}_{IV}	2^{8-4}_{IV}	2^{9-5}_{III}	2^{10-6}_{III}	2^{11-7}_{III}
32				2^{5}	2^{6-1}_{VI}	2^{7-2}_{IV}	2^{8-3}_{IV}	2^{9-4}_{IV}	2^{10-5}_{IV}	2^{11-6}_{IV}
64					2^{6}	2^{7-1}_{VII}	2^{8-2}_{V}	2^{9-3}_{IV}	2^{10-4}_{IV}	2^{11-5}_{IV}
128						2^{7}	2^{8-1}_{VIII}	2^{9-2}_{VI}	2^{10-3}_{V}	2^{11-4}_{V}
256							2^{8}	2^{9-1}_{IX}	2^{10-2}_{VI}	2^{11-3}_{VI}
512								2^{9}	2^{10-1}_{X}	2^{11-2}_{VII}

Runs	12	13	14	15	16	17	18	19	20
4									
8									
16	2^{12-8}_{III}	2^{13-9}_{III}	2^{14-10}_{III}	2^{15-11}_{III}					
32	2^{12-7}_{IV}	2^{13-8}_{IV}	2^{14-9}_{IV}	2^{15-10}_{IV}	2^{16-11}_{IV}	2^{17-12}_{III}	2^{18-13}_{III}	2^{19-14}_{III}	2^{20-15}_{III}
64	2^{12-6}_{IV}	2^{13-7}_{IV}	2^{14-8}_{IV}	2^{15-9}_{IV}	2^{16-10}_{IV}	2^{17-11}_{IV}	2^{18-12}_{IV}	2^{19-13}_{IV}	2^{20-14}_{IV}
128	2^{12-5}_{IV}	2^{13-6}_{IV}	2^{14-7}_{IV}	2^{15-8}_{IV}	2^{16-9}_{IV}	2^{17-10}_{IV}	2^{18-11}_{IV}	2^{19-12}_{IV}	2^{20-13}_{IV}
256	2^{12-4}_{VI}	2^{13-5}_{V}	2^{14-6}_{V}	2^{15-7}_{V}	2^{16-8}_{V}	2^{17-9}_{V}	2^{18-10}_{IV}	2^{19-11}_{IV}	2^{20-12}_{IV}
512	2^{12-3}_{VI}	2^{13-4}_{VI}	2^{14-5}_{VI}	2^{15-6}_{VI}	2^{16-7}_{VI}	2^{17-8}_{VI}	2^{18-9}_{VI}	2^{19-10}_{V}	2^{20-11}_{V}

Full Factorial Design — Resolution V (or Higher) Design — Resolution IV Design — Resolution III Design

13.5.2 Comparing fractions of equal resolution: aberration

There may be many designs of maximum resolution for a given problem.

Aberration as an additional index that can be used to „break ties“ among designs of equal resolution. The aberration of a design is the number of words of shortest length (or effects of lowest order) that are aliased with „I“ in the generator.

E.g., consider two 2_{IV}^{7-2} fractional factorial designs specified by:

$$\mathrm{I} = +\mathrm{ABCD} = +\mathrm{DEFG} = +\mathrm{ABCEFG}$$
$$\mathrm{I} = +\mathrm{ABCD} = +\mathrm{CDEFG} = +\mathrm{ABEFG}$$

The second has less aberration because it aliases a single four-factor interaction with „I“ while the first aliases two four-factor interactions. In general, the goal is to find a design of:

1. maximum resolution (maximum length of shortest word), and among these
2. minimum aberration (minimum number of shortest words)

13.6 Blocking regular fractional factorial designs

The basic idea of regular blocking for fractional factorials is the same as in complete experiments, i.e. we select one or more factorial contrasts to be confounded with blocks.

Inferences about these contrasts can only be made when blocks are modeled as random effects.

The difference to complete experiments is that the contrasts selected are not individual factorial effects, but the strings of effects aliased because the design is a fraction.

In the above example we have chosen block 2 with generator $I = -AB = +ACD = -BCD$ for the regular fractional factorial plan and got 4 estimable strings:

$$„I“ = I - AB + ACD - BCD \qquad „C“ = C - ABC + AD - BD$$
$$„A“ = A - B + CD - ABCD \qquad „D“ = D - ABD + AC - BC$$

Blocking regular fractional factorial designs (cont.)

The fractional factorial design for the chosen block 2 is

a	bc	bd	acd

It can be divided into two blocks of size 2 by confounding one of the effect strings with blocks.

Suppose we select „D“, i.e. we split the treatments included in the fraction using the D column or any other column associated with an effect in the „D“ alias group. The resulting blocked fractional factorial is then:

a	bc		bd	acd

from which we could estimate the 2 effect strings „A“ and „C“. The contrast associated with „D“ is now confounded with the block difference.

So this design clearly does not support simultaneous estimation of all four main effects.

13.7 Augmenting regular fractional factorial designs
13.7.1 Combining fractions

Experimenters often use fractional factorials in sequence so as to intelligently „build up“ the information needed about the joint effects of the factors.

Suppose, for example, that the 2^{4-2} fractional factorial design above generated with $\mathrm{I} = -\mathrm{AB} = +\mathrm{ACD} = -\mathrm{BCD}$ has been completed.

If the resulting data are interesting, a reasonable reaction is to expand the study - i.e. to *augment* the design:

The initial $\frac{1}{4}$ fraction can be converted to a regular $\frac{1}{2}$ fraction by adding any one of the other $\frac{1}{4}$ fractions based on the same generating relation, e.g.

$$\begin{array}{rl} \text{block 2}: & \mathrm{I} = -\mathrm{AB} = +\mathrm{ACD} = -\mathrm{BCD} \\ +\ \text{block 1}: & \mathrm{I} = +\mathrm{AB} = -\mathrm{ACD} = -\mathrm{BCD} \\ \hline =\ \text{combined } \frac{1}{2} \text{ fraction}: & \mathrm{I} = -\mathrm{BCD} \end{array}$$

Combining fractions (cont.)

The combined $\frac{1}{2}$ fraction $\mathrm{I} = -\mathrm{BCD}$ is of resolution III. We could have selected a different second $\frac{1}{4}$ fraction, e.g.

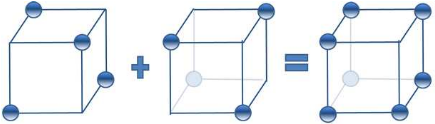
$$\begin{array}{ll} \quad \text{block 2 :} & \mathrm{I} = -\mathrm{AB} = +\mathrm{ACD} = -\mathrm{BCD} \\ + \text{ block 3 :} & \mathrm{I} = -\mathrm{AB} = -\mathrm{ACD} = +\mathrm{BCD} \\ \hline = \text{combined } \frac{1}{2} \text{ fraction :} & \mathrm{I} = -\mathrm{AB} \end{array}$$

this combined $\frac{1}{2}$ fraction $\mathrm{I} = -\mathrm{AB}$ is just of resolution II, so the combined fraction $\mathrm{I} = -\mathrm{BCD}$ would ordinarily be preferred.

If the second fraction is identified by reversing the signs of the „smallest“ factors (here AB), we may increase the resolution of the combined experiment.

Often we would have the opportunity to use a half-fraction of greater resolution then the combination of quarter-fractions. In our example above the half-fraction $\mathrm{I} = +\mathrm{ABCD}$ has resolution IV. Hence there can be an „information cost“ associated with sequential experimentation.

13.7.2 Fold-over designs



There are two practical techniques for augmenting a resolution III design:

- Imagine a 2_{III}^{6-3} fraction ($\frac{1}{8}$) with generator

$$\mathrm{I} = +\mathrm{ABC} = +\mathrm{CDE} = +\mathrm{ABDE} = -\mathrm{ADF} = -\mathrm{BCDF} = -\mathrm{ACEF} = -\mathrm{BEF}$$

  and that we want more information about factorial effects that involve factor A. We may select a second 2^{6-3} augmenting fraction reversing the sign for only factor A, i.e., add a second stage defined by:

$$\mathrm{I} = -\mathrm{ABC} = +\mathrm{CDE} = -\mathrm{ABDE} = +\mathrm{ADF} = -\mathrm{BCDF} = +\mathrm{ACEF} = -\mathrm{BEF}$$

Fold-over designs (cont.)

The above two 2^{6-3} fractions form a regular 2^{6-2} fraction

$$\mathrm{I} = +\mathrm{CDE} = -\mathrm{BCDF} = -\mathrm{BEF}$$

The aliases of the main effect for A are

$$\mathrm{A} = +\mathrm{ACDE} = -\mathrm{ABCDF} = -\mathrm{ABEF}$$

Also all two-factor interactions involving A are aliased with no other two-factor interaction, e.g.

$$\mathrm{AB} = +\mathrm{ABCDE} = -\mathrm{ACDF} = -\mathrm{AEF}$$

So factor A and all two-factor interactions involving A are individually estimable if there are no interactions of order 3 or higher. This means that the resulting design is effectively of resolution V for factor A (although not for all factors).

Fold-over designs (cont.)

- Suppose again the above one-eighth fraction

$$\mathrm{I} = +\mathrm{ABC} = +\mathrm{CDE} = +\mathrm{ABDE} = -\mathrm{ADF} = -\mathrm{BCDF} = -\mathrm{ACEF} = -\mathrm{BEF}$$

And suppose that that *all* factors are potentially interesting. If we reverse the signs of all factors in the augmenting fraction, the generating relation for the second one-eighth fraction is

$$\mathrm{I} = -\mathrm{ABC} = -\mathrm{CDE} = +\mathrm{ABDE} = +\mathrm{ADF} = -\mathrm{BCDF} = -\mathrm{ACEF} = +\mathrm{BEF}$$

The two 2^{6-3} fractions above form a regular 2^{6-2} fraction

$$\mathrm{I} = +\mathrm{ABDE} = -\mathrm{BCDF} = -\mathrm{ACEF}$$

So the design has been improved to resolution IV (for all factors)

13.7.3 Blocking combined fractions

Combining related fractions can also be viewed as a blocked design. Potential systematic differences during each stage of the experiment may suggest that the analysis of the combined data should account for possible block effects.

Consider a 2^{5-2} design in which two related fractions are used in sequence, but now considered as blocks:

$$\begin{aligned}\text{block 1}: \qquad & \mathrm{I} = +\mathrm{ABC} = -\mathrm{ADE} = -\mathrm{BCDE} \\ \text{block 2}: \qquad & \mathrm{I} = -\mathrm{ABC} = +\mathrm{ADE} = -\mathrm{BCDE}\end{aligned}$$

Taken together, the combined fractional factorial design is defined by

$$\mathrm{I} = -\mathrm{BCDE}$$

If we regard the combined experiment as blocked the effects I and $-\mathrm{BCDE}$ are not only aliased with each other, but they are also confounded with (or „sacrificed to“) blocks.

Blocking combined fractions (cont.)

The remaining alias groups of effects are orthogonal to blocks, and so the associated effect strings, e.g. „ABC“ $= \text{ABC} - \text{ADE}$ can be estimated.

When block effects are assumed to be random, information about the factorial effects confounded with blocks can be „recovered“ through inter-block analysis.

Such split-plot analyses can be applied to blocked fractional factorial plans also, where effect strings confounded with blocks are analyzed at the whole-plot stratum.

13.8 Irregular fractional factorial designs

Regular fractional factorial designs require that the number of treatments included is a power of 2.

Plackett-Burman (1946) designs are resolution III, two-level fractional factorial designs for which first-order effects are orthogonally estimable.

The number of treatments included in the smallest Plackett-Burman design that will accommodate f factors is the smallest multiple of 4 that is at least $f + 1$. Larger designs can be constructed for any larger value of n that is also a multiple of 4.

Plackett-Burman designs for which n is not a power of 2 are called *nongeometric* designs, they provide orthogonal estimates of main effects, but the alias structure between these estimates and interactions is more complex than in the regular case.

N	First Row																		
8	+	+	+	−	+	−	−												
12	+	+	−	+	+	+	−	−	−	+	−								
16	+	+	+	+	−	+	−	+	+	−	−	+	−	−	−				
20	+	+	−	−	+	+	+	+	−	+	−	+	−	−	−	−	+	+	−