Advice for difficult homework exercises Experimental Design #5

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summer term 2025

- **ad 17:** The first part must not be a problem at all ...

  For the second part of the exercise we start from the model in the form

  $$\mathbf{y} = \mathbf{1}\mu + \mathbf{X}_\beta \boldsymbol{\beta} + \mathbf{X}_\gamma \boldsymbol{\gamma} + \mathbf{X}_2 \boldsymbol{\tau} + \boldsymbol{\varepsilon}$$

  where

  - $\mathbf{X}_\beta$ is the design matrix corresponding to the b_1 random block effects $\boldsymbol{\beta}$ with $\mathsf{E}(\boldsymbol{\beta}) = \mu_\beta \mathbf{1}_{b_1}$ and $\mathsf{Var}(\boldsymbol{\beta}) = \sigma_\beta^2 \mathbf{I}_{b_1}$ and
  - $\mathbf{X}_\gamma$ is the design matrix corresponding to the b_2 fixed block effects $\boldsymbol{\gamma}$

  we analyze the transformed model for

  $$\begin{aligned}\mathbf{y}_2 = \mathbf{X}_\beta^T \mathbf{y} &= \mathbf{X}_\beta^T \mathbf{1}\mu + \mathbf{X}_\beta^T \mathbf{X}_\beta \boldsymbol{\beta} + \mathbf{X}_\beta^T \mathbf{X}_\gamma \boldsymbol{\gamma} + \mathbf{X}_\beta^T \mathbf{X}_2 \boldsymbol{\tau} + \mathbf{X}_\beta^T \boldsymbol{\varepsilon} = \\ &= b_2 \mu \mathbf{1}_{b_1} + b_2 \boldsymbol{\beta} + \mathbf{1}_{b_2}^T \boldsymbol{\gamma} \mathbf{1}_{b_1} + \mathbf{X}_\beta^T \mathbf{X}_2 \boldsymbol{\tau} + \boldsymbol{\varepsilon}_{i.}\end{aligned}$$

  where

  - $\mathbf{1}_{b_2}^T \boldsymbol{\gamma} \mathbf{1}_{b_1}$ is a b_1-vector where all components are $\sum_{j=1}^{b_2} \gamma_j$ and
  - $\boldsymbol{\varepsilon}_{i.}$ is a b_1-vector where the i-th component is the sum of the b_2 independent errors of row-block i

  Now we separate the mean and the error part of the random effect $\boldsymbol{\beta} = \mu_\beta \mathbf{1}_{b_1} + \boldsymbol{\varepsilon}_\beta$ and combine the intercept and the constant sum of γ-parameters:

  $$\begin{aligned}\mathbf{y}_2 - b_2 \mu_\beta \mathbf{1}_{b_1} &= \mathbf{1}_{b_1} \left(b_2 \mu + \mathbf{1}_{b_2}^T \boldsymbol{\gamma} \right) + \mathbf{X}_\beta^T \mathbf{X}_2 \, \boldsymbol{\tau} + \left(b_2 \boldsymbol{\varepsilon}_\beta + \boldsymbol{\varepsilon}_{i.} \right) = \\ &= \mathbf{Z}_1 \, \nu + \mathbf{Z}_2 \, \boldsymbol{\tau}_{inter} + \boldsymbol{\varepsilon}_2\end{aligned}$$

  i.e.

  - $\nu = b_2 \mu + \mathbf{1}_{b_2}^T \boldsymbol{\gamma}$ is the nuisance parameter for the (now) CRD with corresponding design matrix $\mathbf{Z}_1$ and
  - and $\boldsymbol{\varepsilon}_2$ is the new error term with $\mathsf{E}(\boldsymbol{\varepsilon}_2) = \mathbf{0}_{b_1}$ and $\mathsf{Var}(\boldsymbol{\varepsilon}_2) = (b_2^2 \sigma_\beta^2 + b_2 \sigma^2) \mathbf{I}_{b_1}$

  For this new model go for the estimates of the differences between the treatment parameters ...

  Note that the standard errors for the estimated differences between $\boldsymbol{\tau}_{inter}$ are much higher than these for the differences between $\boldsymbol{\tau}$ in the fixed effects model.

  Why is the following argument wrong? „~~On the other hand the standard errors for the estimated differences between~~ $\cancel{\boldsymbol{\tau}_{intra}}$ ~~are smaller because of the bigger error degrees of freedom of the intra-model.~~