Advice for difficult homework exercises Experimental Design #3

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- **ad 7:**

$$\mathbf{y} = \alpha + \beta_i + \tau_j + \varepsilon_{ij} \qquad i = 1, \dots, b \quad j = 1, \dots, t$$

The answer to (a) was somehow found correctly by everyone who tried the exercise:

$$\mathbf{X}_1 = \left(\begin{array}{ccccc} \mathbf{1}_{t+r} & \mathbf{1}_{t+r} & & & \\ \hline \mathbf{1}_{t+r} & & \mathbf{1}_{t+r} & & \\ \hline \vdots & & & \ddots & \\ \hline \mathbf{1}_{t+r} & & & & \mathbf{1}_{t+r} \end{array} \right) \qquad \mathbf{X}_2 = \left(\begin{array}{cc} \mathbf{1}_r & \mathbf{O} \\ \multicolumn{2}{c}{\mathbf{I}_t} \\ \hline \mathbf{1}_r & \mathbf{O} \\ \multicolumn{2}{c}{\mathbf{I}_t} \\ \hline \multicolumn{2}{c}{\vdots} \\ \hline \mathbf{1}_r & \mathbf{0} \\ \multicolumn{2}{c}{\mathbf{I}_t} \end{array} \right)$$

$\mathbf{X}_1$ is a $(b(t+r) \times (b+1))$-matrix, $\mathbf{X}_2$ is a $(b(t+r) \times t)$-matrix. $\mathbf{X}_1$ has just rank b, $\mathbf{X}_2$ has full rank t

For part (b) we have to find a generalized inverse for

$$\mathbf{X}_1^T \mathbf{X}_1 = \begin{pmatrix} b(t+r) & (t+r)\mathbf{1}_b^T \\ (t+r)\mathbf{1}_b & (t+r)\mathbf{I}_b \end{pmatrix}$$

In general we may find a generalized inverse by setting rows and columns to zero which may be expressed as linear combination of the rest of the rows and columns. In our case we have a $((b+1) \times (b+1))$-matrix $\mathbf{X}_1^T \mathbf{X}_1$ with rank $\mathrm{rk}(\mathbf{X}_1^T \mathbf{X}_1) = b$, i.e. we have to find one row and column that is linear dependent of the rest.

Solution: first row and first column: Now we may easily find generalized inverse - we just leave the zeros and invert the rest:

$$\left(\mathbf{X}_1^T \mathbf{X}_1\right)^{-} = \begin{pmatrix} 0 & \mathbf{o}_b^T \\ \mathbf{o}_b & \frac{1}{t+r}\mathbf{I}_b \end{pmatrix}$$

The rest isn't too difficult:

$$\mathbf{H}_1 = \mathbf{X}_1 (\mathbf{X}_1^T \mathbf{X}_1)^{-} \mathbf{X}_1^T = \frac{1}{t+r} \begin{pmatrix} \mathbf{J}_{t+r} & & & \\ & \mathbf{J}_{t+r} & & \\ & & \ddots & \\ & & & \mathbf{J}_{t+r} \end{pmatrix}$$

$$\mathbf{X}_{2|1} = (\mathbf{I} - \mathbf{H}_1)\mathbf{X}_2 \; = \; \mathbf{X}_2 - \frac{1}{t+r} \left(\begin{array}{cc} (r+1)\mathbf{1}_{b(t+r)} & \mathbf{J}_{b(t+r)\times(t-1)} \end{array} \right)$$

Now it is advantageous to continue with part (e): Show that this design and a CRD with $n_1 = b(r1)$ and $n_j = b, \quad j = 2, \ldots, t$ satisfy condition E!

What are the consequences of condition E?

Yes! We may use the formulas for the equivalent CRD!

Now solve parts (c) and (d) ...

For part (f) use the estimators in following form:

$$\hat{\sigma}^2 = \mathrm{MSE} = \frac{1}{n - \mathrm{rk}(\mathbf{X})} \, \mathbf{y}^T(\mathbf{I} - \mathbf{H})\mathbf{y} =$$

$$= \frac{1}{b(t+r) - bt} \, \mathbf{y}^T(\mathbf{I} - \mathbf{H})\mathbf{y} = \frac{1}{br} \, \mathbf{y}^T(\mathbf{I} - \mathbf{H})\mathbf{y}$$

$$\widehat{\mathbf{c}^T\boldsymbol{\tau}} = \mathbf{c}^T(\mathbf{X}_{2|1}^T\mathbf{X}_{2|1})^{-}\mathbf{X}_{2|1}^T\mathbf{y}$$

and there exists a vector $\mathbf{l}$ s.t. $\mathbf{c}^T = \mathbf{l}^T\mathbf{X}_{2|1}$. Consequently

$$\widehat{\mathbf{c}^T\boldsymbol{\tau}} = \mathbf{l}^T\mathbf{H}_{2|1}\mathbf{y}$$

Now show that $\widehat{\mathbf{c}^T\boldsymbol{\tau}}$ and $\frac{1}{\sqrt{br}}\mathbf{y}^T(\mathbf{I} - \mathbf{H})$ are independent because then also $\widehat{\mathbf{c}^T\boldsymbol{\tau}}$ and $\hat{\sigma}^2$ are independent (why?). Don't try to compute the matrices $\mathbf{H}_{2|1}$ and $\mathbf{H}$ explicitly but think about what the linear spaces are that $\mathbf{H}_{2|1}$ and $\mathbf{H}$ project on and how these spaces are related.