7 Operators

Goal: Instead of doing everything separately for degs and recs, design a uniform theory that covers both cases (and possibly others)

Idea: View a deq $p_0 y + \dots + p_r y^{(r)} = 0$ as an operator equation $L \cdot y = 0$ for $L = p_0 \,\mathrm{id} + p_1 D + \dots + p_r D^r$, and a rec $p_0(x) y(x) + \dots + p_r(x) y(x+r) = 0$ as operator eq $L \cdot y = 0$ with $L = p_0 \,\mathrm{id} + p_1 S + \dots + p_r S^r$.

Operators are just polynomials in a new variable (D or S above). They act on a "function space", e.g.

$$C(x)[D] \times \mathcal{F} \longrightarrow \mathcal{F}.$$

Typically, operators are "nice" and functions are "ugly". We want to do computations in the nice domain and conclude something about the ugly domain.

<u>Want:</u> $\mathcal{F}$ should be a $C(x)$-vector space with a C-linear map $\partial: \mathcal{F} \to \mathcal{F}$ such that $\partial(pf) = p'f + p\,\partial(f)$ for all $p \in C(x)$, $f \in \mathcal{F}$ (and slightly differently for the shift case). We can then define

$$(p_0 + p_1 D + \dots + p_r D^r) \cdot f := p_0 f + \dots + p_r \partial^r(f).$$

We then have

$$L \cdot (f + g) = L \cdot f + L \cdot g$$
$$(L + M) \cdot f = L \cdot f + M \cdot f$$

for all operators L, M and all functions f, g.

We also want: $(LM)\cdot f = L\cdot(M\cdot f)$ for all $L, M \in \mathbb{C}(x)[D]$ and all $f \in \mathcal{F}$.

<u>Problem:</u> We are asking for too much:

$$D\cdot(xf) = f + xf' \overset{!}{=} (Dx)\cdot f$$
$$= (xD)f \overset{!}{=} x(Df)$$
$$= xf' \quad ⚡$$

<u>Solution:</u> give up commutativity of multiplication of operators in favor of $(LM)f = L(Mf)$. Keep in mind that operator multiplication is some sort of composition (like matrix multiplication), so it is not too surprising that we loose commutativity.

<u>Def:</u> Let R be a ring and $\sigma: R \to R$ be an automorphism.

(1) A map $\delta : R \to R$ is called a σ-derivation on R if

$$\forall a,b \in R: \quad \delta(a+b) = \delta(a) + \delta(b)$$
$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

(2) Let $\delta : R \to R$ be a σ-derivation. The ring $R[\partial]$ together with the unique (possibly non-commutative) multiplication satisfying

$$\partial a = \sigma(a)\partial + \delta(a) \qquad \forall a \in R$$

is called the <u>Ore algebra</u> over R for σ and δ.

<u>Ex:</u>

(1) $R = C(x)$, $\sigma = \mathrm{id}$, $\delta = \frac{d}{dx}$ (differential operators)

(2) $R = C(x)$, $\sigma : x \rightsquigarrow x+1$, $\delta = 0$ (recurrence operators)

(3) $R = C(x)$, $\sigma : x \rightsquigarrow x+1$, $\delta = \sigma - \mathrm{id}$ (difference ops)

(4) $R = C(q)(x)$, $\sigma : x \rightsquigarrow qx$, $\delta = 0$ ("q-rec operators")

(5) $R = C(x)$, $\sigma : x \rightsquigarrow x^k$ ($k \in \mathbb{N}$ fix), $\delta = 0$

("Mahler operators")

(6) $R = C(x)$, $\sigma = \mathrm{id}$, $\delta = x\frac{d}{dx}$

(Euler differential operators)

⋮

<u>Ex:</u> In $C(x)[D]$ with $\sigma = \mathrm{id}$, $\delta = \frac{d}{dx}$

we have

$$
\begin{aligned}
&(a + b\partial)(c + d\partial) \\
&= ac + ad\partial + b\partial c + b\partial d\partial \\
&= ac + ad\partial + b(c\partial + c') + b(d\partial + d')\partial \\
&= ac + ad\partial + b(c\partial + c') + bd\partial^2 \\
&= (ac + bc') + (ad + bc + bd')\partial + bd\partial^2
\end{aligned}
$$

<u>Note:</u> Powers of ∂ can always be brought to the right, using the "commutation rule" $\partial a = \sigma(a)\partial + \delta(a)$.

<u>Def:</u> Let $R[\partial]$ be an Ore algebra over R for σ and δ. Let $\mathcal{F}$ be an R-module and $d: \mathcal{F} \to \mathcal{F}$ be such that $d(f+g) = d(f) + d(g)$ for all $f, g \in \mathcal{F}$ and

$d(pf) = \delta(p)f + \sigma(p)d(f)$ for all $p \in R$ and $f \in \mathcal{F}$. Then $\mathcal{F}$ together with the action $R[\partial] \times \mathcal{F} \to \mathcal{F}$,

$$(p_0 + \dots + p_r\partial^r) \cdot f = p_0 f + \dots + p_r d^r(f)$$

is called an $R[\partial]$-module (or ∂-module for short).

<u>Ex:</u> $R = C[x][\partial]$ with $\sigma = \mathrm{id}$, $\delta = \frac{d}{dx}$
$\mathcal{F} = C[[x]]$.

<u>Def:</u> Let $R[\partial]$ be an Ore algebra and $\mathcal{F}$ be a ∂-module.

(1) For $f \in \mathcal{F}$ the set

$$\mathrm{ann}(f) := \{ L \in R[\partial] \mid L \cdot f = 0 \} \subseteq R[\partial]$$

is called the <u>annihilator</u> of f

(2) For $L \in R[\partial]$, the set

$$V(L) := \{ f \in \mathcal{F} \mid L \cdot f = 0 \} \subseteq \mathcal{F}$$

is called the <u>solution space</u> of L.

Facts:

(1) $\mathrm{ann}(f)$ is a left ideal of $R[\partial]$.

The quotient module $R[\partial]/\mathrm{ann}(f)$ is equivalent to the sub-∂-module generated by f in $\mathcal{F}$.

More precisely, if R is a field, the $R[\partial]/\mathrm{ann}(f)$ is isomorphic to the vector space generated by f and all its "derivatives" $\partial \cdot f$, $\partial^2 \cdot f$, ... in $\mathcal{F}$.

(2) Define $C := \{ c \in R \mid \forall f \in \mathcal{F} : \partial \cdot (cf) = c\partial(f) \}$ as the set of constants of $\mathcal{F}$.

Then C is a subring of R, in typical examples it's a field.

Then $V(L)$ is a C-vector space.

Thm: An Ore algebra $K[\partial]$ over a field K is a left-Euclidean domain. In particular, it is a left-principal-ideal domain and it has no zero divisors.

The greatest common right divisor of $L_1, L_2 \in K[\partial]$ is defined as an operator $G \in K[\partial]$ of largest possible order (i.e. degree wrt ∂) such that $L_1 = M_1 G$ and $L_2 = M_2 G$ for some $M_1, M_2 \in K[\partial]$. It is uniquely determined by L_1, L_2 up to left-multiplication by nonzero elements of K. The unique monic greatest common right divisor of L_1, L_2 is denoted by $\mathrm{gcrd}(L_1, L_2)$. It can be computed by a straightforward adaption of the Euclidean algorithm. Also the extended Euclidean algorithm carries over to $K[\partial]$. In particular, for all L_1, L_2 there exist M_1, M_2 such that

$$\gcrd(L_1, L_2) = M_1 L_1 + M_2 L_2.$$

<u>Thm:</u> Let $L_1, L_2 \in K[\partial]$. Then:

(1) $\langle L_1 \rangle + \langle L_2 \rangle = \langle \gcrd(L_1, L_2) \rangle$

(2) $V(\gcrd(L_1, L_2)) = V(L_1) \cap V(L_2)$.

<u>Proof:</u> (1) clear. (2) „$\subseteq$" $f \in V(G)$

$\Rightarrow$ $G \cdot f = 0$ $\Rightarrow$ $\underbrace{M_1 G}_{= L_1} f = \underbrace{M_2 G}_{= L_2} f = 0$

„$\supseteq$" $f \in V(L_1) \cap V(L_2)$

$\Rightarrow$ $L_1 f = L_2 f = 0$

$\Rightarrow$ $\cancel{M}$ $G f = (M_1 L_1 + M_2 L_2) f$ $\boxtimes$

The least common left multiple of $L_1, L_2 \in K[\partial]$ is defined as an operator L of minimal order such that $L = M_1 L_1 = M_2 L_2$ for some $M_1, M_2 \in K[\partial] \setminus \{0\}$. It is uniquely determined up to left multiplication by nonzero elements of K. The unique monic least common left multiple of L_1, L_2 is denoted by $\mathrm{lclm}(L_1, L_2)$.

<u>Thm:</u> (1) $\langle L_1 \rangle \cap \langle L_2 \rangle = \langle \mathrm{lclm}(L_1, L_2) \rangle$

(2) $V(\mathrm{lclm}(L_1, L_2)) \supseteq V(L_1) + V(L_2)$.

<u>Proof:</u> (1) clear; (2) Let $f_1 \in V(L_1)$, $f_2 \in V(L_2)$. Then $L_1 f_1 = L_2 f_2 = 0$

$$\Rightarrow \underbrace{M_1 L_1}_{=L} f_1 = \underbrace{M_2 L_2}_{=L} f_2 = 0$$

$$\Rightarrow L \cdot (f_1 + f_2) = 0 \Rightarrow f_1 + f_2 \in \text{lhs}. \quad \blacksquare$$

<u>Remarks:</u>

(1) The inclusion $\subseteq$ only holds if F is sufficiently well-behaved. In practice, it is usually fair to assume that this is the case.

(2) lclm realises the closure property "addition". This should not be confused with operator addition. There are further operations for other closure properties, e.g. the "symmetric product" for "times" when F is a ring.

(3) If M is a right divisor of L (viz L is a left multiple of M) then $V(M) \subseteq V(L)$.
Conversely, if f is some solution of L, then the generator of ann(f) is a right factor of L.
In particular, hg/hexp sols of operators correspond to first order right factors.