Def: A D-finite object is called hyperexponential / hypergeometric iff it satisfies a deq / rec of order 1

Ex: e^x, x^α are hexp, 2^n, $n!$, n^2 are hg.

Note:

(1) The def leaves open where the objects live. Depending on the context, a hexp object may be a "function", a "series", or a "formal object" (or even something different). Recall the concept of differential fields: If $K = C(x,y)$ with $D(c)=0\ \forall c\in C$, $D(x)=1$, $D(y) = \frac{5x^2-3x+5}{7x^2-9x+2}y$, then $y\in K$ is hexp.

(2) First order eqns are easy to solve:

$$p_0 y + p_1 y' = 0 \quad \cdots \quad y = \exp\left(-\int \frac{p_0}{p_1}\right)$$

$$= \underbrace{\exp(\text{rat})}_{\in C(x)} \cdot \underbrace{\prod}_{\text{finite}} \underbrace{\text{poly}}_{\in C[x]}{}^{\overbrace{c}^{\in C}}$$

$p_0(x)\,y(x) + p_1(x)\,y(x+1) = 0 \quad \ldots \quad y(x) = y(0) \prod_{k=0}^{x-1} \left(-\frac{p_0(k)}{p_1(k)}\right)$

$$= c^x\, \Gamma(x-\xi_1)^{e_1} \cdots \Gamma(x-\xi_k)^{e_k}$$

$\in \mathbb{C} \qquad \in \mathbb{Z}$

(3) The product of hexp/hg objects is hexp/hg, but the sum in general is not. However: If y_1, y_2 are hexp/hg then $y_1 + y_2$ is hexp/hg $\Leftrightarrow$ $\frac{y_1}{y_2} = \text{rat}$ (Proof: Ex.)

We say that y_1, y_2 are <u>similar</u> if $y_1/y_2 = \text{rat}$.

Note: For any $p_0, \ldots, p_r \in \mathbb{C}[x]$, the hexp term y is similar to $p_0 y + \ldots + p_r y^{(r)}$, likewise for the shift case.

<u>Goal</u>: Given a deq/rec, find all its hexp/hg solutions.

<u>Differential case:</u> Strategy: find finitely many candidates for the part exp(rat) and for each of them, find the corresponding parts $\prod \text{poly}^c$ (if there are any)

<u>Observe:</u>

(1) If exp(rat) appears in a hexp sol and ξ is a pole of rat, then the expansion of exp(rat) at ξ has the form

$$\exp\left(\frac{c_k}{(x-\xi)^k} + \dots + \frac{c_1}{(x-\xi)}\right) \cdot (x-\xi)^\alpha (1 + \cdots)$$

where $c_1, \dots, c_k \in \mathbb{C}$ are such that

$$\text{rat} - \sum_{i=1}^{k} \frac{c_i}{(x-\xi)^i} \text{ has no pole at } \xi.$$

We will see this part of rat in the exponential parts of the series sols at ξ. Since every rat fun admits a partial fraction decomposition

$$\underbrace{\sum_{i=1}^{k} \frac{c_i}{(x-\xi)^i}}_{\substack{\text{polar part} \\ \text{at } \xi}} + \underbrace{\sum_{j=1}^{n} \frac{u_j}{(x-\varphi)^j}}_{\substack{\text{polar part} \\ \text{at } \varphi}} + \dots + \underbrace{\sum_{\ell=1}^{m} \frac{v_\ell}{(x-\phi)^\ell}}_{\substack{\text{polar part} \\ \text{at } \phi}} + \underbrace{\text{poly}(x)}_{\substack{\text{"polar part} \\ \text{at } \infty\text{"}}}$$

we can find all possible parts exp(rat) by going through all combinations of the "local" exponential parts at the singularities (including ∞).

<u>Ex:</u> $\xi = 0$: $\exp(\frac{1}{x})$, $\exp(\frac{1}{x^2} - \frac{1}{x})$

$\xi = 1$: $\exp(\frac{1}{(1-x)^2})$, 1

$\xi = \infty$: $\exp(x)$

$\Rightarrow$ The only possible choices are

$$\exp\left(\frac{1}{x} + \frac{1}{(1-x)^2} + x\right), \quad \exp\left(\frac{1}{x} + x\right)$$

$$\exp\left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{(1-x)^2} + x\right), \quad \exp\left(\frac{1}{x^2} - \frac{1}{x} + x\right).$$

Note that there are always only finitely many combinations. (The number can be huge though).

(2) For a given candidate exp(rat), set $f = \exp(\text{rat})\, g$ for a new unknown function g into the deq and divide by exp(rat) to obtain a deq for g with coeffs in $C(x)$.

Ex: $\boxed{}\, f''(x) + \boxed{}\, f'(x) + \boxed{}\, f(x) = 0$

$$f(x) = e^{x^2} g(x)$$
$$f'(x) = 2x e^{x^2} g(x) + e^{x^2} g'(x)$$
$$f''(x) = \bigcirc\, e^{x^2} g(x) + \bigcirc\, e^{x^2} g'(x) + \bigcirc\, e^{x^2} g''(x)$$

$$\rightarrow \boxed{}\left(\bigcirc\, g(x) + \bigcirc\, g'(x) + \bigcirc\, g''(x)\right)$$
$$+ \boxed{}\left(\bigcirc\, g(x) + \bigcirc\, g'(x)\right)$$
$$+ \boxed{}\left(g(x)\right) = 0.$$

(3) It remains to find sols $g = \prod \text{poly}^c$.
Note that the exponents need not be integers. But if they are not and if ξ is a root of the corresponding poly, then we will have a series solution $(x-\xi)^c(1+\cdots)$, so it suffices to find for each singularity $\xi \in \mathbb{C}$ all the possible starting exponents.
If $\xi_1 \dots \xi_r \in \mathbb{C}$ are the singularities and $\alpha_1, \dots, \alpha_r \in \mathbb{C}$ are any representatives of the $\mathbb{Z}$-classes of the right exponents c, then $g = (x-\xi_1)^{\alpha_1} \cdots (x-\xi_r)^{\alpha_r}\,\text{rat}$ for some rational function $\text{rat} \in \mathbb{C}(x)$.

We can try all combinations (there are again "only" finitely many), and for each combination construct a deq for the rat and solve it.

In summary, we have the following alg:

<u>Input:</u> A deq

<u>Output:</u> A set of hexp sols so that any other hexp sol is a $\mathbb{C}$-linear-combination of them.

(1) find the local exponential parts at all singularities $\xi \in \mathbb{C} \cup \{\infty\}$

(2) for each exp(rat) obtained as a product of ~~the~~ some exp parts at all singularities, do:

(3) Set $f = \exp(\text{rat}) \cdot g$ and construct a deq for g.

(4) find the $\mathbb{Z}$-classes of the exponents of the series solutions $(x-\xi)^{\alpha}(1+\dots)$ at all singularities ξ of the deq for g. Choose a representative of each class.

(5) For each combination $(\alpha_1, \dots, \alpha_k)$, set $g = \underbrace{(x-\xi_1)^{\alpha_1} \cdots (x-\xi_k)^{\alpha_k}}_{=: Q} \cdot h$ and construct a deq for h

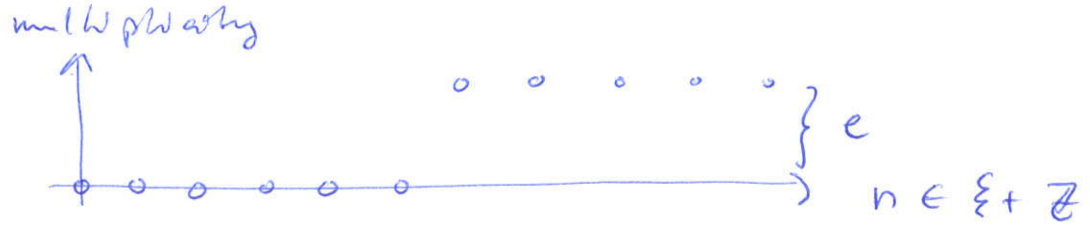
(6) find a basis of the sol space in $C(x)$ of the deq for h. For each basis element b, return $\exp(\text{rat}) Q b$

<u>shift case:</u> The general idea is similar. Recall that every hg term can be written $c^x \prod_i \Gamma(x-\xi_i)^{e_i}$. There may be some cancellation among the Γ-terms. By pulling out a ratfun, we can bring the term to the form $\text{rat}(x)\, c^x \prod_i \Gamma(x-\xi_i)^{e_i}$ with $\xi_j - \xi_i \notin \mathbb{Z}$ $\forall i \neq j$. (eg $\Gamma(x)\Gamma(x+1) = x\,\Gamma(x)^2$). Suppose a given rec has such a sol, and let ξ be one of the ξ_i.

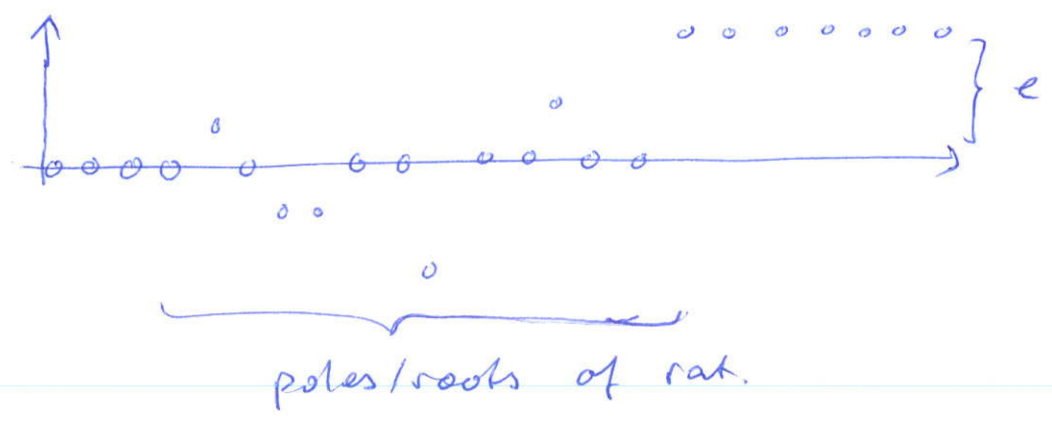
Since $\Gamma(x-\xi)$ satisfies the rec $f(x+1) - (x-\xi) f(x) = 0$, any sol in $C^{\xi+\mathbb{Z}}$ must be zero for all $x > \xi$.

Similarly, $1/\Gamma(x-\xi)$ has the rec $(x-\xi) f(x+1) - f(x) = 0$, of which any sol in $C^{\xi+\mathbb{Z}}$ must be zero for all $x \leq \xi$.

More generally, $\Gamma(x-\xi)^e$ has the rec $f(x+1)-(x-\xi)^e f(x)=0$, and the deformed rec $f(x+1)-(x+q-\xi)f(x)=0$ will have a sol in $\mathbb{C}(q)^{\xi+\mathbb{Z}}$ with a raise/drop of the multiplicity of q in the sequence terms



The deformed rec of $\mathrm{rat}(x)\, c^x \prod_i \Gamma(x-\xi_i)^{e_i}$ will have a sol in $\mathbb{C}(q)^{\xi+\mathbb{Z}}$ with a similar pattern, the only possible difference being caused by poles/roots of rat:



Observations:

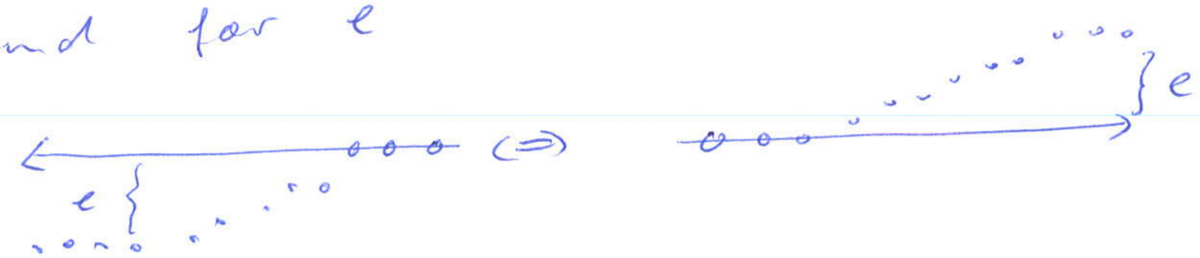
(1) For all ξ_i that may appear in a hg sol, $[\xi_i]$ must be a singularity of the rec (otherwise $e=0$). There are

only finitely many choices.

(2) Each class $[\xi_i]$ contains a ξ_{min} and a ξ_{max} such that the multiplicity of q cannot drop before ξ_{min} or after ξ_{max}. We can extract these from p_0, p_r.

(3) By specifying initials before ξ_{min} and computing sols in $C(q)^{\xi+\mathbb{Z}}$ of the deformed rec beyond ξ_{max}, we can get a lower bound for e

(4) By specifying initials behind ξ_{max} and computing sols in $C(q)^{\xi+\mathbb{Z}}$ of the deformed rec before ξ_{min} (applying the rec backwards), we can get an upper bound for e



(5) Altogether, we find finitely many candidates ξ_i, and for each of them, finitely many candidates for e_i.

For each combination $(e_1, \dots, e_k)$
we can set $f(x) = \prod_i \Gamma(x-\xi_i)^{e_i} \cdot g(x)$
and construct a new rec for $g(x)$.
It remains to find sols of the
form $c^x \operatorname{rat}(x)$ for each of these recs.

(6) What are possible choices for c?
Note that for $\operatorname{rat}(x) = \alpha x^m + \text{l.o.t.}$ we
have $\operatorname{rat}(x+i) = \alpha x^m + \text{l.o.t.}$ for all $i \in \mathbb{N}$
(the shift only affects the l.o.t.).
Therefore, if $d = \max_{i=0}^{r} \deg p_i$, then

$$[x^{d+m}] \sum_{i=0}^{r} p_i \operatorname{rat}(x+i) c^{x+i}$$

$$= \alpha \Big(\sum_{i=0}^{r} [x^d] p_i c^i \Big) c^x = 0 \iff c \text{ is a}$$

root of the poly $\sum_{i=0}^{r} ([x^d]p_i) y^i \in C[y] \setminus \{0\}$.
This leaves finitely many candidates.
Setting $g = c^x h$ reduces the problem
to finding rational sols.