6 Closed form Representations

Task: Given a D-finite function f (specified via deq/rec + initvals), find an "explicit expression" (also called "closed form") which is equal to f, or prove that no such expression exists.

Note: There is no universal definition for which expressions qualify as "closed form". It depends on the context. Some natural choices are:

- polynomials
- rational functions

} today's program

- hypergeometric / hyperexponential fns
- d'Alembertian solutions

} next week's program

- algebraic solutions
- elementary solutions

} not covered in this course.

It is easy to find all polynomial solutions of a prescribed degree d (or less): Just make an ansatz $a_0 + a_1 x + \dots + a_d x^d$ with undetermined coefficients, plug it into the deq/rec, equate coeffs of x^i to zero, and solve the resulting linear system for $a_0 \dots a_d$.

Ex:

(1) $xy' - 2y = 0$ $\qquad y = a_0 + a_1 x + a_2 x^2$

$$x(a_1 + 2a_2 x) - 2(a_0 + a_1 x + a_2 x^2) = 0$$

$$-2a_0 + (a_1 - 2a_1)x + (0)x^2 = 0$$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0$$

$$\ker = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\text{solution space} = \langle x^2 \rangle \subseteq C[x].$$

(2) $(x-1)y(x+1) - (x+1)y(x) = 0$

$$y = a_0 + a_1 x + a_2 x^2$$

$$(x-1)(a_0 + a_1(x+1) + a_2(x+1)^2)$$
$$- (x+1)(a_0 + a_1 x + a_2 x^2) = 0$$

$$(-a_0 - a_1 - a_2 - a_0)$$
$$+ (a_0 - a_1 + a_1 + a_2 - 2a_2 - a_0 - a_1)x$$
$$+ (a_1 + 2a_2 - a_2 - a_1 - a_2)x^2$$
$$+ (0)x^3 = 0$$

$$\begin{pmatrix} -2 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0$$

$$\ker = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

solution space $= \langle x(x-1) \rangle \subseteq \mathbb{C}[x]$.

There are more efficient ways when d is large compared to the size of the equation. For example, in the differential case we can proceed as follows:

(1) compute the associated rec

$q_0(n)a_n + \dots + q_s(n)a_{n+s} = 0$

(2) Find $N \geq d$ such that

$a_N = \dots = a_{N+s-1} = 0 \Rightarrow \forall n \geq N: a_n = 0$

(3) using the rec, compute a basis of the solution space in $\mathbb{C}[[x]]$ of the deq, up to order $N+s$

$$\bigcirc + \bigcirc x + \dots + \bigcirc x^{N+s-1} + \dots$$
$$\bigcirc + \bigcirc x + \dots + \bigcirc x^{N+s-1} + \dots$$
$$\vdots$$
$$\bigcirc + \bigcirc x + \dots + \bigcirc x^{N+s-1} + \dots$$

(4) find the subspace of all fps solutions with coeffs of $x^N \dots x^{N+s-1}$ equal to zero by solving a linear system over $\mathbb{C}$.

For a fixed deq, the cost of this alg is $O(d)$ ops in $\mathbb{C}$.

Note: The set of all polynomial sols of a given equation (regardless of their degree) is a finite dimensional subspace of $C[x]$, since $C[x]$ is an integral domain. It follows that for every deq/rec there exists $d \in \mathbb{N}$ such that all polynomial solutions have degree $\leq d$. (degree bound).

How to find such a degree bound?

(1) differential case. Make an ansatz $x^{\alpha}\underbrace{(1+\cdots)}_{\in C[[x^{-1}]]}$ for a series solution with descending (!) powers of x, plug it into the deq and equate the highest (!) degree term to zero. This gives a polynomial equation with finitely many solutions. The largest integer root is a degree bound.

(2) <u>shift case</u>: It is convenient to write the rec as a difference equation. Define $\Delta f(x) := f(x+1) - f(x)$. Then we can write

$$p_0(x) f(x) + \dots + p_r(x) f(x+r) = 0$$

as
$$q_0(x) f(x) + \dots + q_r(x) \Delta^r f(x) = 0.$$

~~Becau~~ By the binomial theorem, we have $\Delta x^\alpha = (x+1)^\alpha - x^\alpha$

$$= x^\alpha\left(\left(1+\tfrac{1}{x}\right)^\alpha - 1\right) = x^\alpha\Big(\underbrace{\sum_{k=0}^{\infty} \binom{\alpha}{k} x^{-k}}_{= 1 + \alpha x^{-1} + \binom{\alpha}{2} x^{-2} + \dots} - 1\Big)$$

$$= \alpha x^{\alpha-1} + \text{l.o.t.}$$

Therefore, we can proceed as in the differential case.

<u>Ex</u>: $(3x+2) f(x) - (5x^2+1)\Delta f(x) + (2x^3+8x-7)\Delta^2 f(x) = 0$

$$(3x+\dots)(x^\alpha+\dots) - (5x^2+\dots)(\alpha x^{\alpha-1}+\dots) + (2x^3+\dots)(\alpha(\alpha-1)x^{\alpha-2}+\dots) = 0$$

$$\underbrace{(3 - 5\alpha + 2\alpha(\alpha-1))}_{\stackrel{!}{=} 0 \quad \text{no integer roots}} x^{\alpha+1} + \dots = 0 \Rightarrow \text{no poly solutions.}$$

If we want to know whether a specific D-finite function or sequence is a polynomial, we have to check whether it can be written as a linear combination of the basis elements of the solution space in $\mathbb{C}[x]$.

__Ex:__ $(a_n)_{n=0}^{\infty}$ is defined by a certain rec of order 3 and initvals $a_0=1$, $a_1=3$, $a_2=-1$. The solution space of the rec in $\mathbb{C}[x]$ is $\langle x^2+x-1, 3x^3+5\rangle$. To see if (a_n) is a polynomial sequence, make an ansatz:

$$a_n = \alpha(n^2+n-1) + \beta(3x^3+5)$$

$$\left.\begin{array}{ll} n=0 & 1 = -\alpha + 5\beta \\ n=1 & 3 = \alpha + 8\beta \\ n=2 & -1 = 5\alpha + 29\beta \end{array}\right\} \text{ no solution}$$

$\Downarrow$

(a_n) is not a polynomial sequence.

<u>Rational solutions</u> The solution space of a given deq/rec in $C(x)$ has finite dimension. Therefore all the rational sols of a fixed eq share a finite common denominator (the lcm of the denominators of the basis elements of the solution space in $C(x)$; note that C-linear-combinations cannot produce new poles)

<u>Idea:</u> given a deq/rec, construct a poly $Q \in C[x] \setminus \{0\}$ such that for every rational solution $f \in C(x)$ we have $Qf \in C[x]$ (<u>denominator bound</u>). If we have such a Q, we can make a change of variables $f = \frac{g}{Q}$ with g a new unknown function. The rational solutions of the equation for f are exactly the $\frac{1}{Q}$-fold of the polynomial solutions of the equation for g.

<u>differential case:</u> For any rational solution $f \in C(x)$ and any $\xi \in C$, the Laurent series expansion of f at ξ must be a series solution. Therefore $x-\xi$ can only be a factor of the denominator of a rational solution if ξ is a singularity of the equation. Furthermore, in this case, computing the series solutions $(x-\xi)^{\alpha}(1 + \text{hot})$ gives a bound on the multiplicity of each factor. If C is algebraically closed, this gives a denominator bound.

<u>Ex:</u> $(x-1)(x-3)f''(x) + \bigcirc f'(x) + \bigcirc f(x) = 0$

$\xi = 1$ .... $(x-1)^{-3}(1 + \bullet(x-1) + \bullet(x-1)^2 + \cdots)$
$(x-1)^{1/2}(1 + \bullet(x-1) + \bullet(x-1)^2 + \cdots)$

$\xi = 3$ .... $(x-3)^{-2}(1 + \cdots)$
$(x-3)^{-1}(1 + \cdots)$

$\Rightarrow$ $Q = (x-1)^3(x-3)^2$ is a denominator bound.

<u>shift case:</u> This is more tricky. Consider a rec and a rational solution $f \in C(x)$. Suppose that $\xi \in C$ is a pole of f. Since f can have at most finitely many poles, there must be some "rightmost" and "leftmost" poles of f in the equivalence class $\xi + \mathbb{Z}$. Call them ξ_{min}, ξ_{max}. Then the leftmost and rightmost poles of $f(x+i)$ are $\xi_{min} - i$ and $\xi_{max} - i$, respectively. Because of

$$p_r(x) f(x+r) = -p_0(x) f(x) - \dots - p_{r-1}(x) f(x+r-1)$$

it follows that $x - (\xi_{min} - r) \mid p_r$ (because $x - (\xi_{min} - r)$ is part of the denominator of $f(x+r)$ but not of the denominator of the right hand side). Similarly, we must have $x - \xi_{max} \mid p_0$.

Poles of rational solutions can ~~thus~~ only appear in equivalence classes $\xi + \mathbb{Z}$ which contain at least one root of p_r and at least one root of p_0. For each such class, we can get candidates for ξ_{min} and ξ_{max} by inspecting p_r and p_0, respectively.

Each such pair (ξ_{min}, ξ_{max}) can contribute some factors

$$\prod_{i=0}^{\xi_{max}-\xi_{min}} \left(x - (\xi_{min}+i)\right)^{e_i}$$

to the denominator bound. It remains to find the multiplicities e_i.

One way of doing so is to consider a deformed recurrence

$$p_0(x+q)f(x) + \cdots + p_r(x+q)f(x+r) = 0$$

with q a fresh variable. Note that $f(x)$ is a solution of this rec iff $f(x-q)$ is a solution of the original recurrence. Also note that $p_r(x+q) \in C(q)[x]$ has no integer roots.

We can specify initial values $f_i(x-\xi_{min}-j-1) = \delta_{ij}$ and compute the following $\xi_{max}-\xi_{min}$ terms of these solutions of the deformed equation. It turns out that we can take

$$e_i = \max_{k=0}^{r-1} \{ \text{multiplicity of } q \text{ in the denominator of } f_k(\xi_{min}+i) \in C(q) \}.$$