Symbolic solutions of algebraic differential equations

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Abstract

Consider an algebraic ordinary differential equation (AODE), i.e. a polynomial relation between the unknown function and its derivatives. This polynomial defines an algebraic hypersurface. By considering rational parametrizations of this hypersurface, we can decide the rational solvability of the given AODE, and in fact compute the general rational solution. This method depends crucially on curve and surface parametrization and the determination of rational invariant algebraic curves.

Transforming the ambient space by some group of transformations, we get a classification of AODEs, such that equivalent equations share the property of rational solvability. In particular we discuss affine and birational transformation groups.

We also discuss the extension of this method to non-rational parametrizations and solutions.

This research has been carried out jointly with
L.X.Châu Ngô, J.Rafael Sendra, and Georg Grasegger.

Outline

The problem

Rational parametrizations

The autonomous case

The general (non-autonomous) case

Classification of AODEs / differential orbits

Extension to non-rational solutions

Conclusion

The problem

An algebraic ordinary differential equation (AODE) is given by

$$F(x, y, y', \ldots, y^{(n)}) = 0 \ ,$$

where F is a differential polynomial in $K[x]\{y\}$ with K being a differential field and the derivation $'$ being $\frac{d}{dx}$.
Such an AODE is autonomous iff $F \in K\{y\}$.

The radical differential ideal $\{F\}$ can be decomposed

$$\{F\} \;=\; \underbrace{(\{F\} : S)}_{general\ \ component} \;\cap\; \underbrace{\{F, S\}}_{singular\ \ component} \quad ,$$

where S is the separant of F (derivative of F w.r.t. $y^{(n)}$).
If F is irreducible, $\{F\} : S$ is a prime differential ideal; its generic zero is called a general solution of the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$.

J.F. Ritt, *Differential Algebra* (1950)
E. Hubert, The general solution of an ODE, *Proc. ISSAC 1996*

Problem: Rational general solution of AODE of order 1

given: an AODE $F(x, y, y') = 0$, F irreducible in $\overline{\mathbb{Q}}[x, y, y']$

decide: does this AODE have a rational general solution

find: if so, find it

Example: $F \equiv y'^2 + 3y' - 2y - 3x = 0$.
general solution: $y = \frac{1}{2}((x+c)^2 + 3c)$, where c is an arbitrary constant.
The separant of F is $S = 2y' + 3$. So the singular solution of F is $y = -\frac{3}{2}x - \frac{9}{8}$.

Rational parametrizations

An algebraic variety $\mathcal{V}$ is the zero locus of a (finite) set of polynomials F, or of the ideal $I = \langle F \rangle$.
A rational parametrization of $\mathcal{V}$ is a rational map $\mathcal{P}$ from a full (affine, projective) space covering $\mathcal{V}$; i.e. $\mathcal{V} = \overline{\mathrm{im}(\mathcal{P})}$ (Zariski closure).
A variety having a rational parametrization is called unirational; and rational if $\mathcal{P}$ has a rational inverse.

- a parametrization of a variety is a generic point or generic zero of the variety; i.e. a polynomial vanishes on the variety if and only if it vanishes on this generic point
- so only irreducible varieties can be rational
- a rationally invertible parametrization $\mathcal{P}$ is called a proper parametrization;
  every rational curve or surface has a proper parametrization (Lüroth, Castelnuovo); but not so in higher dimensions

For details on parametrizations of algebraic curves we refer to
J.R. Sendra, F. Winkler, S. Pérez-Díaz,
Rational Algebraic Curves – A Computer Algebra Approach,
Springer-Verlag Heidelberg (2008)

The autonomous case $F(y, y') = 0$

First we concentrate on algebraic and geometric questions:

- A rational solution of $F(y, y') = 0$ corresponds to a proper (because of the degree bounds) rational parametrization of the algebraic curve $F(y, z) = 0$.
- Conversely, from a proper rational parametrization $(f(x), g(x))$ of the curve $F(y, z) = 0$ we get a rational solution of $F(y, y') = 0$ if and only if there is a linear rational function $T(x)$ such that $f(T(x))' = g(T(x))$.
  If $T(x)$ exists, then a rational solution of $F(y, y') = 0$ is: $y = f(T(x))$.
  The rational general solution of $F(y, y') = 0$ is (for an arbitrary constant C): $y = f(T(x + C))$

Feng and Gao described a complete algorithm along these lines

R. Feng, X-S. Gao, "Rational general solutions of algebraic ordinary differential equations", Proc. ISSAC2004. ACM Press, New York, 155-162, 2004.

R. Feng, X-S. Gao, "A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs", J. Symb. Comp., 41, 739-762, 2006.

based on degree bounds derived in

J.R. Sendra, F. Winkler, "Tracing index of rational curve parametrizations", Comp.Aided Geom.Design, 18:771–795, 2001.

The general (non-autonomous) case $F(x, y, y') = 0$

- When we consider the autonomous algebraic differential equation $F(y, y') = 0$, it is necessary that $F(y, z) = 0$ is a rational curve. Otherwise, the differential equation $F(y, y') = 0$ has no non-trivial rational solution.
- It is now natural to assume that the solution surface $F(x, y, z) = 0$ is a rational algebraic surface, i.e. rationally parametrized by

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

  The differential condition on y can now be turned into differential conditions on the parameters s and t. We get the associated system:

$$s' = \frac{f_1(s, t)}{g(s, t)}, \quad t' = \frac{f_2(s, t)}{g(s, t)}. \tag{1}$$

L.X.C. Ngô, F. Winkler, "Rational general solutions of first order non-autonomous parametrizable ODEs", J. Symb. Comp., 45(12), 1426–1441, 2010.

Properties of the associated system:

The associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}$ has the form

$$s' = \frac{N_1(s,t)}{M_1(s,t)}, \quad t' = \frac{N_2(s,t)}{M_2(s,t)} \tag{2}$$

The corresponding polynomial system of (2) is

$$s' = N_1 M_2, \quad t' = N_2 M_1. \tag{3}$$

Theorem

There is a one-to-one correspondence between rational general solutions of the algebraic differential equation $F(x, y, y') = 0$, which is parametrized by $\mathcal{P}(s, t)$, and rational general solutions of its associated system with respect to $\mathcal{P}(s, t)$.

The associated system is

- autonomous
- of order 1
- of degree 1 in the derivatives of the parameters

Solving the associated system

Lemma

Every non-trivial rational solution of the associated system (2) corresponds to a rational algebraic curve $G(s,t)=0$ satisfying

$$G_s \cdot N_1 M_2 + G_t \cdot N_2 M_1 \in \langle G \rangle \ . \tag{4}$$

Definition

A rational algebraic curve $G(s,t)=0$ satisfying (4) is called a rational invariant algebraic curve of the system (2).

In case the system (2), (3) has no dicritical singularities, i.e., in the generic case, there is an upper bound for irreducible invariant algebraic curves:
M.M. Carnicer, "The Poincaré problem in the nondicritical case", Annals of Mathematics, 140(2):289–294, 1994.

Reparametrization:

Theorem

Let $G(s,t) = 0$ *be a rational invariant algebraic curve of the associated system* (2) *such that* $G \nmid M_1$ *and* $G \nmid M_2$*. Let* $(s(x), t(x))$ *be a proper rational parametrization of* $G(s,t) = 0$*. W.l.o.g. assume* $s'(x) \neq 0$*.*
Then $(s(x), t(x))$ *creates a rational solution of the associated system if and only if there is a linear rational function* $T(x)$ *such that*

$$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T), t(T))}{M_1(s(T), t(T))}. \tag{5}$$

In this case, $(s(T(x)), t(T(x)))$ *is a rational solution of the associated system.*

L.X.C. Ngô, F. Winkler, "Rational general solutions of planar rational systems of autonomous ODEs", J. Symb. Comp. 46(10), 1173–1186, 2011.

Rational general solutions

Invariant algebraic curves come in families depending on parameters. Such families give rise to rational general solutions.

Theorem

Let $\mathcal{R}(x) = (s(x), t(x))$ be a non-trivial rational solution of the system (2). Let $H(s, t)$ be the monic defining polynomial of the curve parametrized by $\mathcal{R}(x)$.
Then $\mathcal{R}(x)$ is a rational general solution of the system (2) if and only if the coefficients of $H(s, t)$ contain a transcendental constant.

Example: L.X.C. Ngô, F. Winkler, "Rational general solutions of parametrizable AODEs", Publ.Math.Debrecen, 79(3–4), 573–587, 2011.
Consider the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0 \ .$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ has the parametrization

$$\mathcal{P}(s, t) = \left(\frac{t}{s} + \frac{2s + t^2}{s^2}, -\frac{1}{s} - \frac{2s + t^2}{s^2}, \frac{t}{s}\right) .$$

This is a proper parametrization and its associated system is

$$s' = st, \quad t' = s + t^2 \ .$$

Irreducible invariant algebraic curves of the system are:

$$G(s, t) = s, \quad G(s, t) = t^2 + 2s, \quad G(s, t) = s^2 + ct^2 + 2cs$$

The third algebraic curve $s^2 + ct^2 + 2cs = 0$ depends on a transcendental parameter c. It can be parametrized by

$$\mathcal{Q}(x) = \left(-\frac{2c}{1+cx^2}, -\frac{2cx}{1+cx^2} \right).$$

Running Step 5 in RATSOLVE, the differential equation defining the reparametrization is $T' = 1$. Hence $T(x) = x$. So the rational solution in this case is

$$s(x) = -\frac{2c}{1+cx^2}, \quad t(x) = -\frac{2cx}{1+cx^2}.$$

Since $G(s,t)$ contains a transcendental constant, the above solution is a rational general solution of the associated system. Therefore, the rational general solution of $F(x, y, y') = 0$ is

$$y = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c},$$

which, after a change of parameter, can be written as

$$y = \frac{1}{2}(x^2 + 2cx + c^2 + 3c).$$

Classification of AODEs / differential orbits

- consider a group of transformations leaving the associated system of an AODE invariant; orbits w.r.t. such a transformation group contain AODEs of equal complexity in terms of determining rational solutions
- we study some well-known classes of equations and relate them to this algebro-geometric approach
- it turns out that being autonomous is not an intrinsic property of an AODE; certain classes contain both autonomous and non-autonomous AODEs

Affine transformations

L.X.C. Ngô, J.R. Sendra, F. Winkler, "Classification of algebraic ODEs with respect to their rational solvability", Contemporary Mathematics 572, 193–210 (2012)

The group $\mathcal{G}$ of affine transformations

$$\begin{array}{rccc} L: & \mathbb{A}^3(\mathbb{K}) & \longrightarrow & \mathbb{A}^3(\mathbb{K}) \\ & v & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix} v + \begin{pmatrix} 0 \\ c \\ b \end{pmatrix} \end{array}$$

leaves the associated system of an AODE invariant, and therefore also the rational solvability.

Theorem

The group $\mathcal{G}$ defines a group action on AODEs by

$$\begin{array}{rcl} \mathcal{G} \times \mathcal{AODE} & \rightarrow & \mathcal{AODE} \\ (L, F) & \mapsto & L \cdot F = (F \circ L^{-1})(x, y, y')\ . \end{array}$$

Theorem

Let F be a parametrizable AODE, and $L \in \mathcal{G}$. For every proper rational parametrization $\mathcal{P}$ of the surface $F(x, y, z) = 0$, the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(L \cdot F)(x, y, y') = 0$ w.r.t. $L \circ \mathcal{P}$ are equal.

Example: As in the previous example we consider the differential equation

$$F(x,y,y') \equiv y'^2 + 3y' - 2y - 3x = 0 \ .$$

We first check whether in the class of F there exists an autonomous AODE. For this, we apply a generic L to F to get

$$(L\cdot F)(x,y,y') = \frac{1}{a^2}y'^2 + \frac{3}{a}y' - \frac{2b}{a^2}y' - \frac{2}{a}y + \frac{2b}{a}x - 3x - \frac{3b}{a} + \frac{b^2}{a^2} + \frac{2c}{a}.$$

Therefore, for every $a \neq 0$ and b such that $2b - 3a = 0$, we get an autonomous AODE. In particular, for $a = 1$, $b = 3/2$, and $c = 0$ we get

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix} \right],$$

i.e., we obtain

$$F(L^{-1}(x,y,y')) \equiv y'^2 - 2y - \frac{9}{4} = 0 \ .$$

Birational transformations

The group $\mathcal{G}$ of birational transformations from $\mathbb{K}^3$ to $\mathbb{K}^3$ of the form

$$\Phi(u_1, u_2, u_3) = \left(u_1, \quad \frac{au_2 + b}{cu_2 + d}, \quad \frac{\partial}{\partial u_1}\left(\frac{au_2 + b}{cu_2 + d}\right) + \frac{\partial}{\partial u_2}\left(\frac{au_2 + b}{cu_2 + d}\right) \cdot u_3 \right),$$

where $a, b, c, d \in \mathbb{K}[u_1]$ such that $ad - bc \neq 0$,
defines a group action on $\mathcal{AODE}$ by

$$\Phi \cdot F = (F \circ \Phi^{-1})(x, y, y').$$

These birational transformations leave the associated system of an AODE invariant, and therefore also the rational solvability.

We call such a transformation solution preserving.

Problem:

given: $F(x, y, y') \in \mathcal{AODE}$,

decice: does there exist a solution preserving transformation Φ s.t. $G = \Phi \cdot F$ is autonomous?
And, if so, can we compute such a Φ (and therefore G) ?

Example: Consider the first order AODE

$$\begin{aligned} F(x,y,y') = \quad & 25x^2y'^2 - 50xyy' + 25y^2 + 12y^4 - 76xy^3 + \\ & 168x^2y^2 - 144x^3y + 32x^4 \qquad = 0. \end{aligned}$$

Using the transformation

$$\Phi(u,v,w) = \left(u,\ \frac{u-3v}{-2u+v},\ \frac{-5v}{(2u-v)^2} + \frac{5u}{(2u-v)^2}w\right)$$

we get the autonomous equation

$$G(y,y') = F(\Phi^{-1}(x,y,y')) = y'^2 - 4y = 0.$$

Observe that F cannot be transformed into an autonomous AODE by affine transformations.

The rational general solution $y = (x+c)^2$ of $G(y,y') = 0$ is transformed into the rational general solution of $F(x,y,y') = 0$:

$$y = \frac{x(2(x+c)^2+1)}{(x+c)^2+3}.$$

Extension to non-rational solutions

results by G. Grasegger (my PhD student)

Suppose y is a solution of the autonomous AODE $F(y, y') = 0$.
Then $P_y = (y(t), y'(t))$ is a parametrization of the solution surface $F(y, z) = 0$.
For any parametrization $P = (r(t), s(t))$ of the solution surface we consider $A_P = s(t)/r'(t)$.
Assume the parametrization is of the form
$P_g = (r(t), s(t)) = (y(g(t)), y'(g(t)))$, for unknown y and g.
If we could find g, and its inverse g^{-1}, we also could find y:

$$A_{P_g} = \cdots = \frac{1}{g'(t)}$$

$$\text{So } g'(t) = \frac{1}{A_{P_g}}, \quad g(t) = \int g'(t)dt, \quad y(x) = r(g^{-1}(x))$$

we might determine a solution if we can compute the integral and the inverse

Examples:

(a) $y^8 y' - y^5 - y' = 0$:
parametrization: $(\frac{1}{t}, \frac{t^3}{1-t^8})$, $\quad g(t) = \frac{1+t^8}{4t^4}$,
radical solution: $y(x) = -\left(2(x+c) - \sqrt{-1+4(x+c)^2}\right)^{-1/4}$

(b) $4y^7 - 4y^5 - y^3 - 2y' - 8y^2 y' + 8y^4 y' + 8yy'^2 = 0$: (genus 1)
parametrization: $\left(\frac{1}{t}, \frac{-4+4t^2+t^4}{t(4t^2-4t^4-t^6-\sqrt{t^{12}+8t^{10}+16t^8-16t^4})}\right)$
radical solution: $y(x) = -\frac{\sqrt{1+c+x}}{\sqrt{1+(c+x)^2}}$

(c) $y^3 + y^2 + y'^2 = 0$:
parametrization: $(-1-t^2, t(-1-t^2))$, $\quad g(t) = 2\arctan(t)$,
trigonometric solution: $y(x) = -1 - \tan\left(\frac{x+c}{2}\right)^2$

(d) $y^2 + y'^2 + 2yy' + y = 0$:
parametrization: $\left(-\frac{1}{(1+t)^2}, -\frac{t}{(1+t)^2}\right)$
$g(t) = -2\log(t) + 2\log(1+t)$
exponential solution: $y(x) = -e^{-x}(-1+e^{x/2})^2$

Conclusion

- we can decide whether an AODE has rational solutions; and if it has, we can determine the general rational solution
- we have a characterization of the affine and birational transformations of the ambient space leaving the rational solvability of AODEs invariant; this leads (sometimes) to a simplification of the equation
- we have a general method for determining whether an autonomous AODE has a solution in a given class of functions (rational, radical, transcendental); the method depends on the solvability of the problems of integration and inversion in the class of functions; however, this is not a complete method

Thank you for your attention!