VL Formal Modeling (SS 2021)

Symbolic Summation and the modeling of sequences

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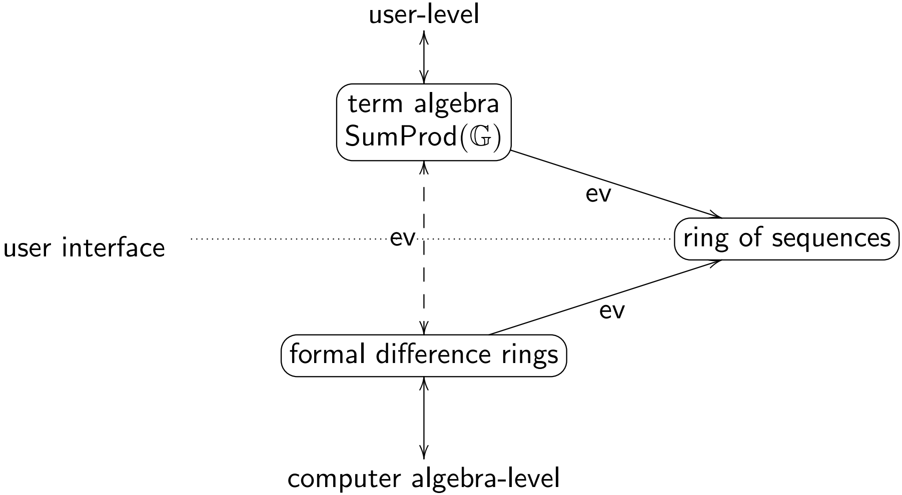
General picture:



General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

The ground field (throughout this talk): $\mathbb{G} = \mathbb{K}(x)$

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- For any element $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x]$ where $q \neq 0$ and p, q being coprime we define

$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0. \end{cases}$$

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- We define $L(f)$ to be the minimal value $\delta \in \mathbb{N}$ such that $q(k) \neq 0$ holds for all $k \geq \delta$; further,

$$Z(f) = \max(L(\tfrac{1}{p}), L(\tfrac{1}{q})) \qquad \text{if } f \neq 0.$$

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Example: For

$$f = \frac{p}{q} = \frac{x-4}{(x-3)(x-1)}$$

we get

$$(\operatorname{ev}(f,n))_{n\geq 0} = (-\tfrac{4}{3}, \underline{0}, 2, \underline{0}, 0, \tfrac{1}{8}, \dots) \in \mathbb{Q}^{\mathbb{N}}$$

For $n \geq L(f) = 4$ no poles arise;

for $n \geq Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) = \max(4,5) = 5$ no zeroes arise.

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- We define

$$\mathcal{R} = \{r \in \mathbb{K} \setminus \{1\} \mid r \text{ is a root of unity}\}$$

with the function $\operatorname{ord} : \mathcal{R} \to \mathbb{Z}_{\geq 1}$ where

$$\operatorname{ord}(r) = \min\{n \in \mathbb{Z}_{\geq 1} \mid r^n = 1\}.$$

$\mathbb{G} \quad\longrightarrow\quad$ SumProd($\mathbb{G}$) (nested sums over hypergeometric products)

Let $\owedge$, $\oplus$, $\odot$, Sum, Prod and RPow be operations with the signatures

$$\begin{array}{llll}
\owedge: & \mathrm{SumProd}(\mathbb{G})\times\mathbb{Z} & \to & \mathrm{SumProd}(\mathbb{G})\\
\oplus: & \mathrm{SumProd}(\mathbb{G})\times\mathrm{SumProd}(\mathbb{G}) & \to & \mathrm{SumProd}(\mathbb{G})\\
\odot: & \mathrm{SumProd}(\mathbb{G})\times\mathrm{SumProd}(\mathbb{G}) & \to & \mathrm{SumProd}(\mathbb{G})\\
\mathrm{Sum}: & \mathbb{N}\times\mathrm{SumProd}(\mathbb{G}) & \to & \mathrm{SumProd}(\mathbb{G})\\
\mathrm{Prod}: & \mathbb{N}\times\mathrm{SumProd}(\mathbb{G}) & \to & \mathrm{SumProd}(\mathbb{G})\\
\mathrm{RPow}: & \mathcal{R} & \to & \mathrm{SumProd}(\mathbb{G}).
\end{array}$$

Prod$^*(\mathbb{G})$= the smallest set that contains 1 with the following properties:

1. If $r \in \mathcal{R}$ then RPow$(r) \in$ Prod$^*(\mathbb{G})$.
2. If $f \in \mathbb{G}^*$ and $l \in \mathbb{N}$ with $l \geq Z(f)$ then Prod$(l, f) \in$ Prod$^*(\mathbb{G})$.
3. If $p, q \in$ Prod$^*(\mathbb{G})$ then $p \odot q \in$ Prod$^*(\mathbb{G})$.
4. If $p \in$ Prod$^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p^{\owedge} z \in$ Prod$^*(\mathbb{G})$.

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\text{Sum}: & \mathbb{N} \times \text{SumProd}(\mathbb{G}) & \to \text{SumProd}(\mathbb{G}) \\
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Furthermore, we define

$$\Pi(\mathbb{G}) = \{\text{RPow}(r) \mid r \in \mathcal{R}\} \cup \{\text{Prod}(l, f) \mid f \in \mathbb{G}, l \in \mathbb{N}\}.$$

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Example: In $\mathbb{G} = \mathbb{Q}(x)$ we get

$$P = (\underbrace{\mathsf{Prod}(1, x)}_{\in \Pi(\mathbb{G})}{}^{\circledwedge}(-2)) \odot \underbrace{\mathsf{RPow}(-1)}_{\Pi(\mathbb{G})} \in \mathsf{Prod}^*(\mathbb{G}).$$

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SumProd($\mathbb{G}$) = the smallest set containing $\mathbb{G} \cup \mathsf{Prod}^*(\mathbb{G})$ with:

1. For all $f, g \in \mathsf{SumProd}(\mathbb{G})$ we have $f \oplus g \in \mathsf{SumProd}(\mathbb{G})$.
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Furthermore, the **set of nested sums over hypergeometric products** is given by

$$\Sigma(\mathbb{G}) = \{\mathsf{Sum}(l, f) \mid l \in \mathbb{N} \text{ and } f \in \mathsf{SumProd}(\mathbb{G})\}$$

and the **set of nested sums and hypergeometric products** is given by

$$\Sigma\Pi(\mathbb{G}) = \Sigma(\mathbb{G}) \cup \Pi(\mathbb{G}).$$

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Example

With $\mathbb{G} = \mathbb{K}(x)$ we get, e.g., the following expressions:

$$E_1 = \text{Sum}(1, \text{Prod}(1, x)) \in \Sigma(\mathbb{G}) \subset \text{SumProd}(\mathbb{G}),$$

$$E_2 = \text{Sum}(1, \tfrac{1}{x+1} \odot \text{Sum}(1, \tfrac{1}{x^3}) \odot \text{Sum}(1, \tfrac{1}{x})) \in \Sigma(\mathbb{G}) \subset \text{SumProd}(\mathbb{G}),$$

$$E_3 = (E_1 \oplus E_2) \odot E_1 \in \text{SumProd}(\mathbb{G}).$$

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$$\begin{aligned}\mathrm{ev}(f \circledwedge k,n) &:= \mathrm{ev}(f,n)^k,\\ \mathrm{ev}(f\oplus g,n) &:= \mathrm{ev}(f,n)+\mathrm{ev}(g,n),\\ \mathrm{ev}(f\odot g,n) &:= \mathrm{ev}(f,n)\ \mathrm{ev}(g,n);\end{aligned}$$

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\end{aligned}$$

2. for $r \in \mathcal{R}$ and $\mathsf{Sum}(l, f), \mathsf{Prod}(\lambda, g) \in \mathsf{SumProd}(\mathbb{G})$ we define

$$\begin{aligned}
\mathrm{ev}(\mathsf{RPow}(r), n) &:= \prod_{i=1}^{n} r = r^n, \\
\mathrm{ev}(\mathsf{Sum}(l, f), n) &:= \sum_{i=l}^{n} \mathrm{ev}(f, i), \\
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$\mathrm{ev} : \mathbb{G} \times \mathbb{N} \to \mathbb{K} \qquad \longrightarrow \qquad \mathrm{ev} : \mathsf{SumProd}(\mathbb{G}) \times \mathbb{N} \to \mathbb{K}$

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\end{aligned}$$

Note: $\Pi(\mathbb{G})$ defines all hypergeometric products (which evaluate to sequences with non-zero entries).

ev applied to $f \in \mathsf{SumProd}(\mathbb{G})$ represents a sequence.
f can be considered as a simple program and $\mathrm{ev}(f, n)$ with $n \in \mathbb{N}$ executes it (like an interpreter/compiler) yielding the nth entry of the represented sequence.

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Example

For $E_i \in \mathsf{SumProd}(\mathbb{K}(x))$ with $i = 1, 2, 3$ we get

$$E_1(n) = \mathrm{ev}(E_1, n) = \mathrm{ev}(\mathsf{Sum}(1, \mathsf{Prod}(1, x)), n) = \sum_{k=1}^{n} \prod_{i=1}^{k} i = \sum_{k=1}^{n} k!,$$

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$$E_2(n) = \mathrm{ev}(\mathsf{Sum}(1, \tfrac{1}{x+1} \odot \mathsf{Sum}(1, \tfrac{1}{x^3}) \odot \mathsf{Sum}(1, \tfrac{1}{x})), n)$$
$$= \sum_{k=1}^{n} \tfrac{1}{1+k} \Big(\textstyle\sum_{i=1}^{k} \tfrac{1}{i^3} \Big) \textstyle\sum_{i=1}^{k} \tfrac{1}{i}$$

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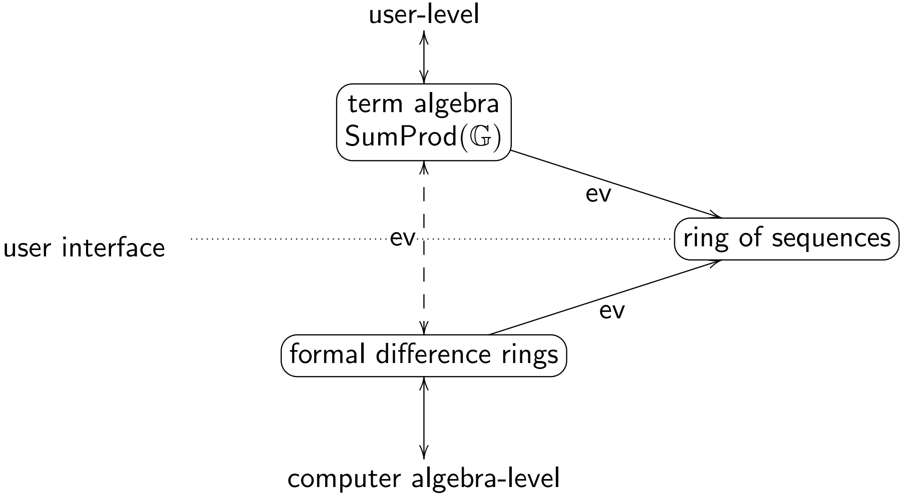
&= \sum_{k=1}^{n} \tfrac{1}{1+k}\Big(\sum_{i=1}^{k}\tfrac{1}{i^3}\Big)\sum_{i=1}^{k}\tfrac{1}{i}\\
E_3(n) &= (E_1(n)+E_2(n))E_1(n)
\end{aligned}
$$

General picture:



Definition

An expression $A \in \mathrm{SumProd}(\mathbb{G})$ is in **reduced representation** if

$$A = (f_1 \odot P_1) \oplus (f_2 \odot P_2) \oplus \cdots \oplus (f_r \odot P_r) \tag{1}$$

with $f_i \in \mathbb{G}^*$ and

$$P_i = (a_{i,1}{}^{\owedge} z_{i,1}) \odot (a_{i,2}{}^{\owedge} z_{i,2}) \odot \cdots \odot (a_{i,n_i}{}^{\owedge} z_{i,n_i})$$

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for $1 \leq i \leq r$ where

- $a_{i,j} = \mathsf{Sum}(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in \mathsf{SumProd}(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z}_{\geq 1}$,
- $a_{i,j} = \mathsf{Prod}(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in \mathsf{Prod}^*(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z} \setminus \{0\}$,
- $a_{i,j} = \mathsf{RPow}(f_{i,j})$ with $f_{i,j} \in \mathcal{R}$ and $1 \leq z_{i,j} < \operatorname{ord}(r_{i,j})$

such that the following properties hold:

1. for each $1 \leq i \leq r$ and $1 \leq j < j' < n_i$ we have $a_{i,j} \neq a_{i,j'}$;
2. for each $1 \leq i < i' \leq r$ with $n_i = n_j$ there does not exist a $\sigma \in S_{n_i}$ with $P_{i'} = (a_{i,\sigma(1)} {}^{\owedge} z_{i,\sigma(1)}) \odot (a_{i,\sigma(2)} {}^{\owedge} z_{i,\sigma(2)}) \odot \cdots \odot (a_{i,\sigma(n_i)} {}^{\owedge} z_{i,\sigma(n_i)})$.

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$H \in \mathsf{SumProd}(\mathbb{G})$ is in **sum-product reduced representation** if

- it is in reduced representation;
- for each $\mathsf{Sum}(l, A)$ and $\mathsf{Prod}(l, A)$ that occur recursively in H the following holds:
  - A is in reduced representation as given in (1);
  - $l \geq \max(L(f_1), \dots, L(f_r))$ (i.e., no poles occur);
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$\mathsf{Sum}(0, \frac{1}{x})$ is not in sum-product reduced represenation

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$$A = (f_1 \odot P_1) \oplus (f_2 \odot P_2) \oplus \cdots \oplus (f_r \odot P_r) \tag{1}$$

with $f_i \in \mathbb{G}^*$

$H \in \mathsf{SumProd}(\mathbb{G})$ is in **sum-product reduced representation** if

- it is in reduced representation;
- for each $\mathsf{Sum}(l, A)$ and $\mathsf{Prod}(l, A)$ that occur recursively in H the following holds:
  - A is in reduced representation as given in (1);
  - $l \geq \max(L(f_1), \ldots, L(f_r))$ (i.e., no poles occur);
  - the lower bound l is greater than or equal to the lower bounds of the sums and products inside of A.

Example

$E_3 = (E_1 \oplus E_2) \odot E_1$ is not in reduced representation

$\mathsf{Sum}(0, \frac{1}{x})$ is not in sum-product reduced represenation

$\mathsf{Sum}(1, \mathsf{Sum}(2, \frac{1}{x}))$ is not in sum-product reduced represenation

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Lemma

For any $A \in \mathsf{SumProd}(\mathbb{G})$, there is a $B \in \mathsf{SumProd}(\mathbb{G})$ in sum-product reduced representation and $\lambda \in \mathbb{N}$ such that

$$A(n) = B(n) \quad \forall n \geq \lambda.$$

Key-Definitions: Let $W \subseteq \Sigma\Pi(\mathbb{G})$.

SumProd$(W, \mathbb{G})$ = the set of elements from SumProd$(\mathbb{G})$ which are in reduced representation and the arising sums/products are taken from W.

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- W is called **shift-closed over** $\mathbb{G}$ if for any $A \in$ SumProd$(W, \mathbb{G})$, $s \in \mathbb{Z}$ there are $B \in$ SumProd$(W, \mathbb{G})$ and $\delta \in \mathbb{N}$ such that

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$$W \text{ is shift-stable } \quad \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} \quad W \text{ is shift-closed}$$

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- W is called **shift-stable over** $\mathbb{G}$ if for any product or sum in W the multiplicand or summand is built by sums and products from W.
- W is called **canonical reduced over** $\mathbb{G}$ if for any $A, B \in$ SumProd$(W, \mathbb{G})$ with
$$A(n) = B(n) \quad \forall n \geq \delta$$
for some $\delta \in \mathbb{N}$ the following holds: A and B are the same up to permutations of the operands in $\oplus$ and $\odot$.

Definition

$W \subseteq \Sigma\Pi(\mathbb{G})$ is called σ-**reduced over** $\mathbb{G}$ if

1. the elements in W are in sum-product reduced form,
2. W is shift-stable (and thus shift-closed) and
3. W is canonical reduced.

In particular, $A \in \mathrm{SumProd}(W, \mathbb{G})$ is called σ-**reduced (w.r.t.** W**)** if W is σ-reduced over $\mathbb{G}$.

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Problem SigmaReduce: Compute a σ-reduced representation

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.
Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$,
$B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$
such that for all $1 \leq i \leq r$ we get

$$A_i(n) = B_i(n) \quad n \geq \delta_i.$$

- **Canonical representation in term algebras**

$$\begin{tikzcd} A_1 \arrow[d] \\ B_1 \end{tikzcd}$$

in SumProd($\mathbb{G}$)

σ-reduced

$\forall n \geq \delta \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n)$

- **Canonical representation in term algebras**

$$\begin{tikzcd} A_1 \arrow[d] & & A_2 \arrow[d] & \text{in SumProd}(\mathbb{G}) \\ B_1 & \sigma\text{-reduced} & B_2 & \end{tikzcd}$$

$$\forall n \geq \delta \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \qquad\qquad \mathsf{ev}(A_2, n) = \mathsf{ev}(B_2, n)$$

- **Canonical representation in term algebras**

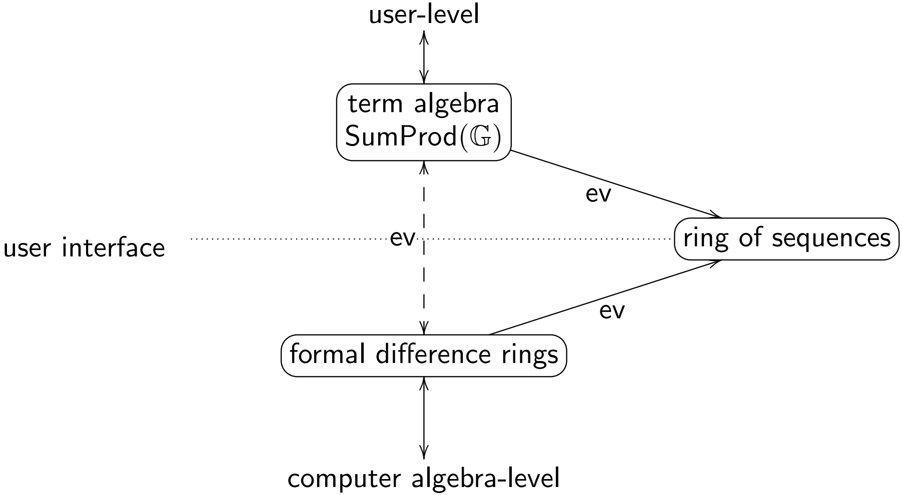
$$\begin{array}{ccc} A_1 & & A_2 \\ \downarrow & & \downarrow \\ B_1 & \sigma\text{-reduced} & B_2 \end{array} \qquad \text{in SumProd}(\mathbb{G})$$

$$\forall n \geq \delta \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \quad \color{blue}{=} \quad \mathsf{ev}(A_2, n) = \mathsf{ev}(B_2, n)$$

$$\color{blue}{\Updownarrow} \qquad \color{blue}{\text{canonical simplifier}}$$

$$\color{blue}{B_1 = B_2}$$

General picture:



General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

Represent $H = \mathsf{Sum}(1, \frac{1}{x}) \in \mathsf{SumProd}(\mathbb{G})$ with

$$H(n) = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

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2. an evaluation function

$$\begin{aligned} \mathrm{ev}' : \quad & \mathbb{Q}(x) \times \mathbb{N} && \to \quad \mathbb{Q} \\ & \left(\tfrac{p(x)}{q(x)}, n\right) && \mapsto \quad \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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$$\begin{array}{rlcl} \mathrm{ev} : & \mathbb{Q}(x)[s] \times \mathbb{N} & \to & \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i\, s^i, n\right) & \mapsto & \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) H_n^i \end{array} \qquad \mathbf{ev(s,n) = H_n}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

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Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \quad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

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It is an injective ring homomorphism (ring embedding):

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Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

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ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = H_n + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

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τ is an injective difference ring homomorphism:

$$\begin{tikzcd}
\mathbb{K}(x)[s] \arrow[r, "\sigma"] \arrow[d, "\tau"] \arrow[dr, phantom, "="] & \mathbb{K}(x)[s] \arrow[d, "\tau"] \\
\mathbb{K}^{\mathbb{N}}/\sim \arrow[r, "S"] & \mathbb{K}^{\mathbb{N}}/\sim
\end{tikzcd}$$

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$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle H_n \rangle_{n \geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}} / \sim, S)$$

Summary: we rephrase $H \in \mathsf{SumProd}(\mathbb{G})$ as element h in a formal difference ring. More precisely, we will design

- a ring $\mathbb{A}$ with $\mathbb{A} \supseteq \mathbb{G} \supseteq \mathbb{K}$ in which H can be represented by $h \in \mathbb{A}$;
- an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ such that $H(n) = \mathrm{ev}(h, n)$ holds for sufficiently large $n \in \mathbb{N}$;
- a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ which models $H(n+1)$ with $\sigma(h)$.

A hypergeometric APS-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

A hypergeometric APS-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x)}$$

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z]}$$

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$$\begin{array}{rcll} & & \sigma(x)=x+1 & \\ \text{hypergeometric} & \leftrightarrow & \sigma(p_1)=a_1\,p_1 & a_1\in\mathbb{K}(x)^* \\ \text{products} & & \sigma(p_2)=a_2p_2 & a_2\in\mathbb{K}(x)^* \\ & & \vdots & \\ & & \sigma(p_e)=a_ep_e & a_e\in\mathbb{K}(x)^* \\ \mathbf{(-1)^k} & \leftrightarrow & \boldsymbol{\sigma(\mathbf{z})=-\mathbf{z}} & \mathbf{z^2=1} \end{array}$$

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$$\begin{array}{rcll} \text{hypergeometric products} & \leftrightarrow & \sigma(p_1) = a_1\, p_1 & a_1 \in \mathbb{K}(x)^* \\ & & \sigma(p_2) = a_2 p_2 & a_2 \in \mathbb{K}(x)^* \\ & & \vdots & \\ & & \sigma(p_e) = a_e p_e & a_e \in \mathbb{K}(x)^* \\ \gamma \text{ is a primitive } \lambda\text{th root of unity} \quad \gamma^{\mathbf{k}} & \leftrightarrow & \boldsymbol{\sigma(\mathbf{z}) = \gamma\, \mathbf{z}} & \mathbf{z}^{\lambda} = \mathbf{1} \end{array}$$

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(nested) sum	$\leftrightarrow$	$\sigma(s_1) = s_1 + f_1$	$f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z]$

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Definition (Evaluation function)

Take $(\mathbb{A},\sigma)$ with a subfield $\mathbb{K}$ of $\mathbb{A}$ with $\sigma|_{\mathbb{K}}=\mathrm{id}$.

1. $\mathrm{ev}:\mathbb{A}\times\mathbb{N}\to\mathbb{K}$ is called **evaluation function** for $(\mathbb{A},\sigma)$ if for all $f,g\in\mathbb{A}$, $c\in\mathbb{K}$ and $l\in\mathbb{Z}$ there exists a $\lambda\in\mathbb{N}$ with

$$\forall n\geq\lambda:\mathrm{ev}(c,n)=c, \tag{2}$$

$$\forall n\geq\lambda:\mathrm{ev}(f+g,n)=\mathrm{ev}(f,n)+\mathrm{ev}(g,n), \tag{3}$$

$$\forall n\geq\lambda:\mathrm{ev}(f\,g,n)=\mathrm{ev}(f,n)\;\mathrm{ev}(g,n), \tag{4}$$

$$\forall n\geq\lambda:\mathrm{ev}(\sigma^{l}(f),n)=\mathrm{ev}(f,n+l). \tag{5}$$

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$$\forall n \geq \lambda : \mathrm{ev}(\sigma^l(f), n) = \mathrm{ev}(f, n + l). \tag{5}$$

2. $L : \mathbb{A} \to \mathbb{N}$ is called o-**function** if for any $f, g \in \mathbb{A}$ with $\lambda = \max(L(f), L(g))$ the properties (3) and (4) hold and for any $f \in \mathbb{A}$ and $l \in \mathbb{Z}$ with $\lambda = L(f) + \max(0, -l)$ property (5) holds.

Connection between SumProd($\mathbb{G}$) and hypergeometric APS-extension

- **Observation 1:** Given $\{T_1, \dots, T_e\} \subseteq \Sigma\Pi(\mathbb{G})$, one can construct a hypergeometric APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with ev and L such that there are $a_1, \dots, a_e \in \mathbb{E}$ and $\delta_1, \dots, \delta_e$ with $\mathrm{ev}(a_i, n) = T_i(n)$.

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- **Observation 2:**

$(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1\rangle \dots \langle t_e\rangle$ a hypergeometric APS-extension of $(\mathbb{G}, \sigma)$ $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$, $L : \mathbb{E} \to \mathbb{N}$

$$\Big\downarrow \qquad \begin{array}{l} \forall n \geq L(t_i) : \\ \mathrm{ev}(t_i, n) = T_i(n) \in \Sigma\Pi(\mathbb{G}) \end{array}$$

$W = \{T_1, \dots, T_e\} \subseteq \Sigma\Pi(\mathbb{G})$ is sum-product reduced and shift stable: sums/products in T_i are from $\{T_1, \dots, T_{i-1}\}$.

In particular, if $f \in \mathbb{E} \setminus \{0\}$, then we can take the "unique" $0 \neq F \in$ SumProd($\{T_1, \dots, T_e\}, \mathbb{G}$) with $F(n) = \mathrm{ev}(f, n)$ for all $n \geq L(f)$.

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Definition

For $f\in\mathbb{E}$ we also write expr$(f)=F$ for this particular F.

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Example

For $f = x + \frac{x+1}{x} s^4 \in \mathbb{Q}(x)[s]$ we obtain

$$\text{expr}(f) = F = x \oplus \left(\frac{x+1}{x} \odot (\text{Sum}(1, \tfrac{1}{x})^{\wedge\!\!\!\bigcirc} 4) \in \text{Sum}(\mathbb{Q}(x))\right)$$

with $F(n) = \mathrm{ev}(f, n)$ for all $n \geq 1$.

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Difference ring theory in action

Let $(\mathbb{E}, \sigma)$ be a hypergeometric APS-extension of $(\mathbb{G}, \sigma)$ with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ and let $\tau : \mathbb{E} \to \mathbb{K}^{\mathbb{N}}/\sim$ be the $\mathbb{K}$-homomorphism given by

$$\tau(f) = (\mathrm{ev}(f, n))_{n \geq 0}.$$

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Lemma

Let $W = \{T_1, \ldots, T_e\} \in \Sigma\Pi(\mathbb{G})$ *with* $T_i =$ *expr*(t_i)*. Then:*

$$W \text{ is canonical reduced} \quad \Leftrightarrow \quad \tau \text{ is injective.}$$

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Let $W = \{T_1, \dots, T_e\} \in \Sigma\Pi(\mathbb{G})$ with $T_i = \mathit{expr}(t_i)$. Then:

$$W \text{ is canonical reduced} \quad \Leftrightarrow \quad \tau \text{ is injective.}$$

Using difference ring theory we get the following crucial property:

Theorem

$$\tau \text{ is injective} \quad \Leftrightarrow \quad \mathrm{const}_\sigma \mathbb{E} = \mathbb{K}.$$

Example

For our difference field $\mathbb{G} = \mathbb{K}(x)$ with $\sigma(x) = x + 1$ and $\mathrm{const}_\sigma \mathbb{K} = \mathbb{K}$ we have $\mathrm{const}_\sigma \mathbb{K}(x) = \mathbb{K}$.

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Definition

A hypergeometric APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ is called **hypergeometric $R\Pi\Sigma$-extension** if

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Theorem

Let $W = \{T_1, \ldots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ be in sum-product reduced representation and shift-stable, i.e., for each $1 \leq i \leq e$ the arising sums and products in T_i are contained in $\{T_1, \ldots, T_{i-1}\}$. Then the following is equivalent:

1. *There is a hypergeometric $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \ldots \langle t_e \rangle$ equipped with an evaluation function ev with $T_i = \mathit{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.*
2. *W is σ-reduced over $\mathbb{G}$.*

This yields a strategy (actually the only strategy for shift-stable sets).

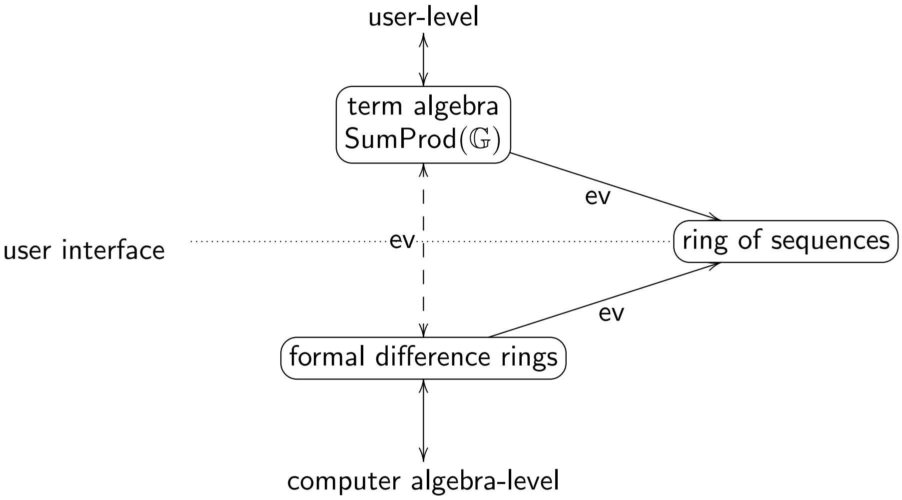
A Strategy to solve Problem SigmaReduce

Given: $A_1,\dots,A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G}=\mathbb{K}(x)$.

Find: a σ-reduced set $W=\{T_1,\dots,T_e\}\subset \Sigma\Pi(\mathbb{G})$ with $B_1\dots,B_u\in \mathsf{SumProd}(W,\mathbb{G})$ and $\delta_1,\dots,\delta_u\in\mathbb{N}$ such that $A_i(n)=B_i(n)$ holds for all $n\geq\delta_i$ and $1\leq i\leq r$.

1. Construct $R\Pi\Sigma$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{G},\sigma)$ with $\mathbb{E}=\mathbb{G}\langle t_1\rangle\dots\langle t_e\rangle$ equipped with $\mathrm{ev}:\mathbb{E}\times\mathbb{N}\to\mathbb{K}$ such that we get $a_1,\dots,a_u\in\mathbb{E}$ and $\delta_1,\dots,\delta_u\in\mathbb{N}$ with
$$A_i(n)=\mathrm{ev}(a_i,n)\quad\forall n\geq\delta_i. \tag{9}$$
2. Set $W=\{T_1,\dots,T_e\}$ with $T_i:=\mathsf{expr}(t_i)\in\Sigma\Pi(\mathbb{G})$ for $1\leq i\leq e$.
3. Set $B_i:=\mathsf{expr}(a_i)\in\mathsf{SumProd}(W,\mathbb{G})$ for $1\leq i\leq u$.
4. Return W, $(B_1,\dots,B_u)$ and $(\delta_1,\dots,\delta_u)$.

General picture:



General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

A hypergeometric APS-extension of $(\mathbb{K}(x), \sigma)$ is

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

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$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)^* \end{aligned}$$

γ is a primitive λth root of unity $\gamma^{\mathbf{k}}$ $\leftrightarrow$ $\sigma(\mathbf{z}) = \gamma\,\mathbf{z}$ $\quad \mathbf{z}^{\lambda} = \mathbf{1}$

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$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

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such that $\mathrm{const}_\sigma\mathbb{E}=\mathbb{K}$

Represent sums (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant set

$$\text{const}_\sigma \mathbb{A} := \{k \in \mathbb{A} | \sigma(k) = k\}.$$

Note 1: $\text{const}_\sigma \mathbb{A}$ is a ring that contains $\mathbb{Q}$

Note 2: We always take care that $\text{const}_\sigma \mathbb{A}$ is a field

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3. $\boxed{\exists g \in \mathbb{A}\setminus\{0\} : \sigma(g) = a^n\,g \text{ only for } n \in \mathbb{Z}\setminus\{0,1\}}$: ☹

The hypergeometric case

- Take the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma|_{\mathbb{K}} = \text{id}$ and $\sigma(x) = x + 1$.
- Let $\alpha_1, \ldots, \alpha_r \in \mathbb{K}(x)^*$

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- $\frac{\sigma(t_i)}{t_i} \in \mathbb{K}(x)^*$ for $1 \leq i \leq e$
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Note: There are similar results for the q-rational, multi-basic and mixed case

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This yields a strategy (actually the only strategy for shift-stable sets).

A Strategy to solve Problem SigmaReduce

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.

Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \dots \langle t_e \rangle$ equipped with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \dots, a_u \in \mathbb{E}$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ with
$$A_i(n) = \mathrm{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{9}$$
2. Set $W = \{T_1, \dots, T_e\}$ with $T_i := \mathsf{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.
3. Set $B_i := \mathsf{expr}(a_i) \in \mathsf{SumProd}(W, \mathbb{G})$ for $1 \leq i \leq u$.
4. Return W, $(B_1, \dots, B_u)$ and $(\delta_1, \dots, \delta_u)$.

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An Algorithm to solve Problem SigmaReduce

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Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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Sigma computes

$$g(k) = (S_1(k) - 1)k.$$

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Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$

$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference ring

FIND a closed form for

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A difference ring for the summand

Consider a ring

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Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

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Polynomial Solution: FIND

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h].$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(x)[h]$

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$$\begin{aligned}&\big[\sigma(g_2\, h^2 + g_1 h + g_0)\big]\\ &\qquad\qquad -\big[g_2\, h^2 + g_1 h + g_0\big] = h\end{aligned}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

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coeff. comp. $\boxed{}$

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$$\sigma(g_1\, h+g_0)-(g_1\, h+g_0)=h-c\Big[\tfrac{2h(x+1)+1}{(x+1)^2}\Big]$$

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g=hx-x

coeff. comp.

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$c = 0,\quad \begin{array}{l} g_1 = x+d \\ d \in \mathbb{Q}\end{array}$

$$\begin{array}{l} g_0 = -x \\ d = 0 \end{array} \;\leftarrow\; \boxed{\sigma(g_0)-g_0 = -1 - d\frac{1}{x+1}}$$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)x \in \mathbb{A}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

Remarks. All results can be generalized to the following setting:

- the mixed multibasic hypergeometric case:
  $\mathbb{G} := \mathbb{K}(x, x_1, \ldots, x_v)$ with $\mathbb{K} = K(q_1, \ldots, q_v)$ For $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x, x_1, \ldots, x_v]$ where $q \neq 0$ and p, q being coprime we define

$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k, q_1^k, \ldots, q_v^k) = 0 \\ \frac{p(k, q_1^k, \ldots, q_v^k)}{q(k, q_1^k, \ldots, q_v^k)} & \text{if } q(k, q_1^k, \ldots, q_v^k) \neq 0. \end{cases}$$

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- simple products: $\mathsf{Prod}^*(\mathbb{G})$ is the smallest set that contains 1 with:

1. If $r \in \mathcal{R}$ then $\mathsf{RPow}(r) \in \mathsf{Prod}^*(\mathbb{G})$.
2. If $f \in \mathbb{G}^*$, $l \in \mathbb{N}$ with $l \geq Z(f)$ then $\mathsf{Prod}(l, f) \in \mathsf{Prod}^*(\mathbb{G})$.
3. If $p, q \in \mathsf{Prod}^*(\mathbb{G})$ then $p \odot q \in \mathsf{Prod}^*(\mathbb{G})$.
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For further details see

Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation. To appear in: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computuation, 2021. Springer, arXiv:2102.01471 [cs.SC]

General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\quad(=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

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FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

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$\uparrow$ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

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Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

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$$= \frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!}$$

$$\underbrace{+\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

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$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

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In[3]:= $\mathbf{res} = \mathbf{SigmaReduce[mySum]}$

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S_1[a]-S_1[a+k]-S_1[a+n]+S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

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$$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

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In[4]:= $\mathbf{SigmaLimit[res, \{n\}, a]}$

Out[4]=
$$\frac{1}{n!}\frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

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Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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no solution ☹

Zeilberger's creative telescoping paradigm

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Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

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Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

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$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n)+c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \quad || \quad -n\mathsf{A}(n)+(2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} \in \boxed{\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} | c \in \mathbb{R}\}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{\begin{aligned} &{\color{blue}0} \times \frac{1}{n(n+1)} \\ &+ \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \end{aligned}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $$\mathbf{mySum} = \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{a}} \frac{\mathbf{S[1,k]+S[1,n]-S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

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Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

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In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-\mathtt{n}\mathsf{SUM}[\mathtt{n}]+(1+\mathtt{n})(2+\mathtt{n})\mathsf{SUM}[\mathtt{n}+1] == \frac{(\mathtt{n}+1)\mathtt{S}[1,\mathtt{n}]+1}{(\mathtt{n}+1)^3}$

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Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{\mathtt{n}(\mathtt{n}+1)}\}, \{1, \frac{\mathtt{S}[1,\mathtt{n}]^2 + \sum_{\mathtt{i}=1}^{\mathtt{n}} \frac{1}{\mathtt{i}^2}}{2\mathtt{n}(\mathtt{n}+1)}\}\}$

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Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[9]= $\frac{\mathtt{S}[1,\mathtt{n}]^2 + \sum_{\mathtt{i}=1}^{\mathtt{n}} \frac{1}{\mathtt{i}^2}}{2\mathtt{n}(\mathtt{n}+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

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$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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FIND all solutions expressible by indefinite nested products/sums
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3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

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$$||$$

$$\boxed{\begin{array}{c}\binom{j+1}{r}\Big(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\\ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_{r}r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_{r}(2-n)_{j}}\Big)\end{array}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

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$$||$$

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$$||$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

> Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

> HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

> EvaluateMultiSums by Carsten Schneider © RISC-Linz

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In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

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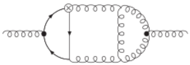
In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

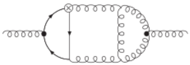
Application: The simplification of Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

Evaluation of Feynman Integrals



Behavior of particles

$$\longrightarrow \quad \int \Phi(N,\epsilon,x)dx$$

Feynman integrals

$$\int_0^1 x\,dx = \text{?}$$

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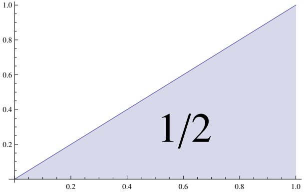
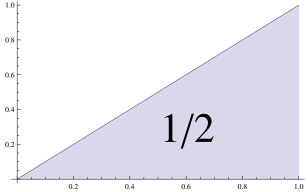
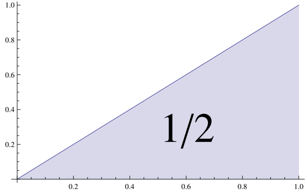


$$\int_0^1 x^1\,dx =$$

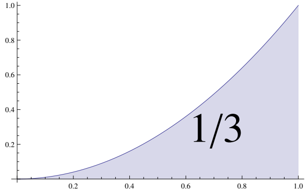


$$\int_0^1 x^2\,dx = \text{?}$$

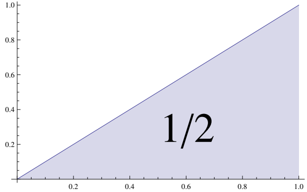
$$\int_0^1 x^1 \, dx =$$



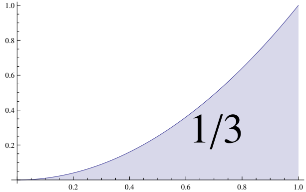
$$\int_0^1 x^2 \, dx =$$



$$\int_0^1 x^1\,dx =$$

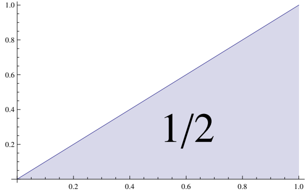


$$\int_0^1 x^2\,dx =$$

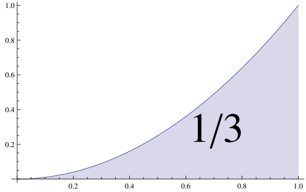


$$\int_0^1 x^3\,dx = ?$$

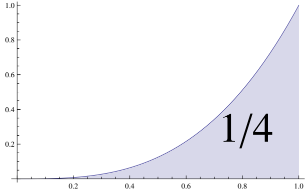
$$\int_0^1 x^1\,dx =$$



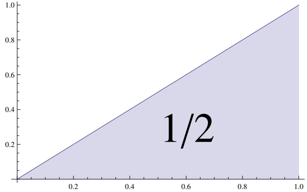
$$\int_0^1 x^2\,dx =$$



$$\int_0^1 x^3\,dx =$$



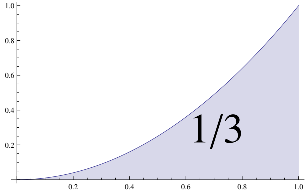
$$\int_0^1 x^1 \, dx =$$



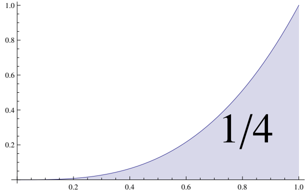
$$\boxed{\int_0^1 x^N \, dx = \frac{1}{N+1}}$$

für $N = 1, 2, 3, 4, \ldots$

$$\int_0^1 x^2 \, dx =$$



$$\int_0^1 x^3 \, dx =$$



Feynman integrals

$$\int_0^1 x^N \, dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N(1+x)^N}{(1-x)^{1+\varepsilon}}\,dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots dx_1\, dx_2\, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

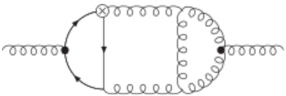
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5\, dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \qquad \Big\|$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

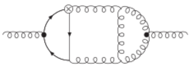
$$\Bigg[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Bigg]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Evaluation of Feynman Integrals

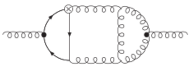


Behavior of particles

$$\longrightarrow \quad \int \Phi(N,\epsilon,x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x) dx$$

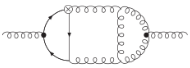
Feynman integrals

DESY
(J. Blümlein)

$$\sum f(N, \epsilon, k)$$

complicated
multi-sums

Evaluation of Feynman Integrals



Behavior of particles $\longrightarrow$ $\int \Phi(N, \epsilon, x) dx$ Feynman integrals

DESY (J. Blümlein) $\downarrow$

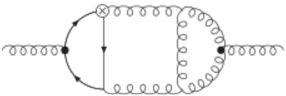
$\sum f(N, \epsilon, k)$ complicated multi-sums

expression in special functions $\xleftarrow[\text{(Sigma-package)}]{\textbf{RISC}}$ complicated multi-sums

Example 1:

massive 3-loop ladder integrals

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

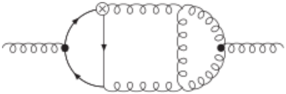
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

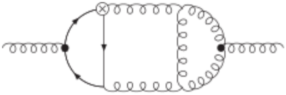
$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\boxed{\begin{aligned}&\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times\\&\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}\\&\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)\\&-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))\\&+2S_1(s-1)-2S_1(r+s)\Big]+\textbf{3 further 6–fold sums}\end{aligned}}$$

$$
\boxed{F_0(N)} =
$$

$$
\begin{aligned}
&\frac{7}{12}S_1(N)^4+\frac{(17N+5)S_1(N)^3}{3N(N+1)}+\Big(\frac{35N^2-2N-5}{2N^2(N+1)^2}+\frac{13S_2(N)}{2}+\frac{5(-1)^N}{2N^2}\Big)S_1(N)^2\\
&+\Big(-\frac{4(13N+5)}{N^2(N+1)^2}+\Big(\frac{4(-1)^N(2N+1)}{N(N+1)}-\frac{13}{N}\Big)S_2(N)+\Big(\frac{29}{3}-(-1)^N\Big)S_3(N)\\
&+\big(2+2(-1)^N\big)S_{2,1}(N)-28S_{-2,1}(N)+\frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N)+\Big(\frac{3}{4}+(-1)^N\Big)S_2(N)^2\\
&-2(-1)^N S_{-2}(N)^2+S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)}+\big(26+4(-1)^N\big)S_1(N)+\frac{4(-1)^N}{N+1}\Big)\\
&+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)}-\frac{5}{2N^2}\Big)S_2(N)+S_{-2}(N)\Big(10S_1(N)^2+\Big(\frac{8(-1)^N(2N+1)}{N(N+1)}\\
&+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N)+\frac{8(-1)^N(3N+1)}{N(N+1)^2}+\big(-22+6(-1)^N\big)S_2(N)-\frac{16}{N(N+1)}\Big)\\
&+\Big(\frac{(-1)^N(9N+5)}{N(N+1)}-\frac{29}{3N}\Big)S_3(N)+\Big(\frac{19}{2}-2(-1)^N\Big)S_4(N)+\big(-6+5(-1)^N\big)S_{-4}(N)\\
&+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)}-\frac{2}{N}\Big)S_{2,1}(N)+\big(20+2(-1)^N\big)S_{2,-2}(N)+\big(-17+13(-1)^N\big)S_{3,1}(N)\\
&-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N)-\big(24+4(-1)^N\big)S_{-3,1}(N)+\big(3-5(-1)^N\big)S_{2,1,1}(N)\\
&+32S_{-2,1,1}(N)+\left(\frac{3}{2}S_1(N)^2-\frac{3S_1(N)}{N}+\frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)
\end{aligned}
$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \big(- \ldots 1)^N(2N+1) \over N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$+ \big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

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$$+ \Big(\cdots 1)^N(2N+1) \over \cdots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \Big(2 + \cdots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 \cdots} \cdots {}^N\Big)S_2(N)^2$$

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$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \Big(-22 + 6(-1)^N\Big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

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$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N} \frac{1}{i^2}$$

$\boxed{F_0(N)} =$

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$$+ \Big(\ldots 1)^N(2N+1) \ldots N(N+1) - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2 + \ldots 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 \ldots} \ldots \big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + (26+4 \ldots\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\big(10S_1(N)^2 + \big(\frac{\ldots(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N \ldots}{N(N+ \ldots} \ldots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\big)$$

$$+ \Big(\frac{(-1)^N \ldots}{N(\ldots} \ldots (N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(- \ldots}{\ldots} \ldots \Big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N \ldots}{N(N+1)} S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

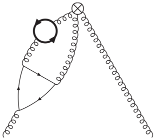
$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i \sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

Example 2:

2-mass 3-loop Feynman integrals

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



All diagrams are produced with axodraw (J. Vermaseren).

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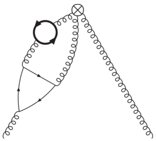
(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow{\text{Mellin-Barnes- and } {}_pF_q\text{-technologies}}$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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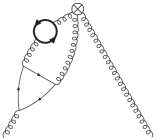


Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

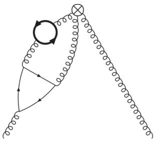
Typical triple sum:

$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon}\times 2^{-2+\varepsilon}e^{-\frac{3\varepsilon\gamma}{2}}\eta^{k}\times$$

$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow{\text{Mellin-Barnes- and } {}_pF_q\text{-technologies}}$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

Typical triple sum:

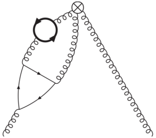
$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon}\times 2^{-2+\varepsilon}e^{-\frac{3\varepsilon\gamma}{2}}\eta^k\times$$
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

6 hours for this sum

$\sim$ 10 years of calculation time for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow[\text{and } {}_pF_q\text{-technologies}]{\text{Mellin-Barnes-}}$

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$\downarrow$ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

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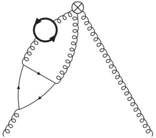

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$\downarrow$ EvaluateMultiSums.m

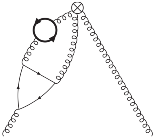
Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation		number of indef. sums
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s	(2.1 days)	1188
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s	(11.4 days)	747
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s	(6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s	(8.0 days)	44
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s	(3.5 days)	1933
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s	(8.2 days)	621
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s	(5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s	(2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow[\text{and } {}_pF_q\text{-technologies}]{\text{Mellin-Barnes-}}$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB)
consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

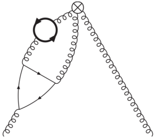
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left(\sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow{\text{Mellin-Barnes- and } {}_pF_q\text{-technologies}}$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

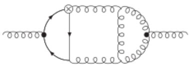
$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

$\xleftarrow{\text{Sigma.m (32 days)}}$

expression (8.3 MB) consisting of 74 indefinite sums

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

DESY
(J. Blümlein)

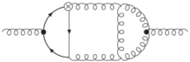
$$\sum f(N,\epsilon,k)$$

complicated
multi-sums

RISC
(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

LHC at CERN

DESY
(J. Blümlein)

$$\sum f(N,\epsilon,k)$$

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expression in special functions

applicable