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Formal Modeling
Simple Quantum Systems

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Abstract

This lecture note contains a brief description of the principles of quantum models and a discussion of two simple quantum system: a qubit and pair of qubits.

1 Fundamentals

A classical deterministic model in physics describes a system as a mathematical object that may assume various states. The current state determines the state in a time in the future by certain physical laws. The state can be observed by performing measurements. The result of such a measurement is a number, the value of some coordinate of the state. By a sufficient number of measurements, the state can be determined, and one can predict the behavior of the system in the future.

There are limitations of such models: Measurements have a only a limited accuracy, so the result cannot be determined exactly. External perturbations that cannot be avoided cause deviations from the predicted behavior. Most importantly: the models just do not work on the microscopic level.

In a classical statistical model, one gives up the notion that we can determine the state exactly. The state still determines the state in a time in the future, but we have only partial information of the state. This does not allow to predict the future exactly, but it still allows to predict probabilities of future events. Measurements only tell average values. Observations do not change the state, but they do increase our knowledge of the state, and make predictions more accurate.

Statistical model make their own limitations very explicit, by specifying precisely what they can afford and what not. There is a tendency that the time development – following the physical laws – decreases the potential of predictabilty. More severely: the models do not work on the microscopic level either.

A quantum model describes a system as a mathematical object that may assume various states, just as the classical deterministic model. Also, the current state determines the state in a time in the future by physical laws (the laws are different, but the principle is the same). However, when a measurement is performed, the quantum model is more like a classical statistical model: the result of a measurement can, in general, not be predicted exactly, we can only predict the probability distribution of the result. But, unlike in a classical statistical model, there is no state that would determine the result of the measurement (and which, in classical statistical models, is just not known to us). In a quantum model, the state determines the probability distribution of any measurement, and there is nothing that would determine the result exactly.

In some sense, we can increase our knowledge by a measurement just as in the classical statistical case: after the result of a measurement, the system is in a state that predicts that a second measurement would give precisely the same result. However, this increase of knowledge comes at the cost of a decrease of knowledge on the result of measuring other observables. The observables of a quantum system are not coordinates of the states: we cannot measure the observables without changing the state, and the determination of the value of an observable A necessily reduces knowledge on the value of an observable B.

In contrast to both types of classical models, the result of a measurement is very often not an arbitary number in some interval, but a number in a discrete set which is known in advance. This fact is responsible for the name. The main advantage of quantum models is that they work perfectly on the microscopic level.

Example: Stern/Gerlach experiments. The *spin* of an electron is similar to the angular momentum: we have a spin in the direction of every unit vector in S^2. The spin values in the direction of opposite vectors sum up to zero. In contrast to classical angular momentum, the spin can only by $\pm\frac{1}{2}$.

To measure the spin, Stern/Gerlach used that the magnetic field of the silver atom depends only on the spin of a single electron. To measure the spin L_z, they shot a ray of silver atoms directed parallel to the y-axis through an external magenrtic field M_x into the direction of the x-axis. Then they measured the spot where the electrons hit a screen parallel to the xz-plane. The observation was that half of the atoms went slightly up, and the other half went slightly down; we see two spots.

Now we block the downray and send the upray throgh a second magnetic field M_z. Half of the upray goes left, half of the upray goes right, and we see again two spots.

Question 1: How many spots would we see when we block the downray and send the upray throogh M_x again? Answer: 1.

Question 2: How many spots would we see when we block the downray and the uprightray and send the upleftray through M_x? Answer: 2.

Question 3: Now we block the downray and send the upray through M_z, and unite the upleftray and the uprightray again using suitable electromagnetic fields, but – this is important – without measuring how many atoms went left and right. The united ray is sent through M_x. How many spots? Answer: 1.

2 The Qubit

A qubit is a quantum system X where the states are vectors of length 1 of a vector space S_X of dimension 2. The basis elements are denoted by $\boxed{0}, \boxed{1}$, so that every state is equal to $\alpha\boxed{0}+\beta\boxed{1}$, where α, β are complex numbers such that $|\alpha|^2+|\beta|^2=1$. After time 1 (we take a discrete time model), the state v is transformed to Uv, where U is a matrix such that $U\bar{U}^t = I_2$. Such a matrix is called a *unitary matrix*. Note that $||Uv|| = ||v||$ for all vectors if U is unitary. Permutation matrizes are always unitary. As far as this lecture is concerned, we may as well assume all coefficients are real and all unitary matrices are real orthogonal matrices.

We have a basic observable of the qubit, with value set $\{0,1\}$. The probablity for 0 is $|\alpha|^2$, and the probability of 1 is $|\beta|^2$. After a result 0, the qubit's state is a multiple of $\boxed{0}$, and after a result 1, the qubit's state is a multiple of $\boxed{1}$. Let us call two states equivalent if one is a scalar multiple o the other (by a scalar of modulus 1, of course). Equivalent states define the same probability distribution, and a unitary transformation preserves equivalence. We do not consider other observables, because they can be simulated by first applying a unitary transformation and then measuring the basic observable.

Qubits can be combined into strings. Let X, Y be two qubits. The states of the system XY are unit vectors of $S_X \otimes S_Y$. A basis for this space is $(\boxed{0}\otimes\boxed{0}, \boxed{0}\otimes\boxed{1}, \boxed{1}\otimes\boxed{0}, \boxed{1}\otimes\boxed{1})$. From now on, we omit the $\otimes$ symbol in vectors. If $U_1, U_2 \in \mathbb{C}^{2\times 2}$ are unitary, then $U_1 \otimes U_2$ is a unitary transformation of $S_X \otimes S_Y$, called the *Kronecker product*.

There are two basis observables: we can measure the first qubit, or we can measure the second qubit. Let v_0 be a state. We can write $v_0 = \alpha\boxed{0}\boxed{0} + \beta\boxed{0}\boxed{1} + \gamma\boxed{1}\boxed{0} + \delta\boxed{1}\boxed{1}$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2 = 1$. The probability of measuring 0 for X is $|\alpha|^2+|\beta|^2$. After having measured zero for X, the state of the system is equivalent to the projection of v_0 to the subspace generated by all vectors of the form $\boxed{0}\otimes w$, $w \in \mathbb{C}^2$. This projection is equal to $v_1 = \frac{\alpha\boxed{0}\boxed{0}+\beta\boxed{0}\boxed{1}}{\sqrt{|\alpha|^2+|\beta|^2}}$. If we now measure Y, then the probabilty of obtaining $\boxed{0}$ is $\frac{|\alpha|^2}{|\alpha|^2+|\beta|^2}$. After the second measurement, the state is equivalent to $\boxed{0}\boxed{0}$. The probability of measuring first $\boxed{0}$ for X and then $\boxed{0}$ for Y is equal to $|\alpha|^2$. Similarly, the probability of measuring first $\boxed{0}$ for X and then $\boxed{1}$ for Y is equal to $|\beta|^2$, etc. If we perform the measurements in a different order, namely first measuring Y and then X, we get exactly the same probability distribution. This is a remarkable relation between the two basic observables, called *commensurable*. Commensurable observable can be simultanously measured, and after the common measurement we have an exact result for both observables.

Let n be a positive integer. Then an n-digit word in the alphabet $\{0,1\}$ can be transformed into a quantum state of the composition of n qubits (by measuring and applying unitary transformations). Afterwards, the system can be transported through some communication channel. On the other side, the partner may measure the quibits one by one and get the transmitted word.

2.1 Entanglement

A state $v = \alpha\boxed{0}\boxed{0} + \beta\boxed{0}\boxed{1} + \gamma\boxed{1}\boxed{0} + \delta\boxed{1}\boxed{1}$ of the system XY is called *entangled* if $\alpha\delta - \beta\gamma \neq 0$. An important example is the *Bell state* $b := \frac{\boxed{0}\boxed{1} - \boxed{1}\boxed{0}}{\sqrt{2}}$. If we measure the two qubits X and Y, then we get 01 or 10, each with probabilty $\frac{1}{2}$. But there is more: if U is an arbitrary unitary matrix, then $(U \otimes U)b$ is equivalent to b. To show this, assume $U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then we get

$$\sqrt{2}(U \otimes U)b = \begin{pmatrix} \alpha^2 & \alpha\beta & \alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta & \beta\gamma & \beta\delta \\ \alpha\gamma & \beta\gamma & \alpha\delta & \beta\delta \\ \gamma^2 & \gamma\delta & \gamma\delta & \delta^2 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha\delta + \beta\gamma \\ -\beta\gamma + \alpha\delta \\ 0 \end{pmatrix} = \sqrt{2}(\alpha\delta - \beta\gamma)b,$$

and the claim is proven.
Recall that measuring an arbitary observable of a qubit can be reduced to applying a unitary matrix and then measuring the basic observable. If a pair of qubits XY in Bell state, then a measurement of any observable for X will always give the opposite value as the measurement of the same observable for Y.
Assume that Alice and Bob share a pair of qubits in Bell state, where Alice controls X and Bob controls Y. Then Alice can transmit two bits to Bob by applying a unitary transformation depending on the two bits, and sending X to Bob. Here is how it works:

00 : Alice applies $C_{00} := I_2$ (i.e., she leaves X as it is).

01 : Alice applies $C_{01} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

10 : Alice applies $C_{10} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

11 : Alice applies $C_{11} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Bob receives X from Alice and applies a certain unitary matrix D to the pair XY and then measures X and Y. For the correct choice of D, the result is equal to the transmitted code word.
To use this method (superdense coding) on a larger scale, Alice and Bob must prepare a sufficient number of pairs of qubits in Bell state together before separating. Then they can

send each other words of length $2n$ by applying this protocol and sending n qubits over the quantum channel.

Exercise 1: Compute $c_{ij} := (C_{ij} \otimes I_2)b$ for $i, j = 0, 1$. Compute a matrix D such that

$$Dc_{00} = \boxed{0|0}, Dc_{01} = \boxed{0|1}, Dc_{10} = \boxed{1|0}, Dc_{11} = \boxed{1|1}.$$

Show that D is unitary. Write a short instruction how to do superdense coding and decoding. Assume that the reader of the instruction is a "quantum engineer" who knows how to apply unitary transformations and do measurements of qubits in his/her control.

Shared pairs of qubits in Bells state is a very valuable resource in quantum computation. If Alice and Bob share a sufficient number of them, then it is possible for Alice to take a collection U of qubits, apply some unitary transformations, do some measurements and send the results to Bob via a classical channel, and then for Bob to do some transformations on his qubits that produces the exact copy of the collection U of qubits Alice has started with. Except it is not really a copy, because the transformations and measurements Alice had to do has messed up her original U. It is as if U would have been physically transported from Alice's place to Bob's place. The process is known as "quantum teleportation".
In order to teleport a single qubit U, Alice and Bob need a single shared pair XY of qubits in Bell state. Let us assume that the qubit to teleport is in the state $u = \alpha\boxed{0} + \beta\boxed{1}$. So, before the teleportation starts, the whole system is in the state

$$u \otimes b = \frac{1}{\sqrt{2}}\left(\alpha\boxed{0|0|1} - \alpha\boxed{0|1|0} + \beta\boxed{1|0|1} - \beta\boxed{1|1|0}\right).$$

Alice controls the first two qubits: she applies the matrix D from Exercise 1 to them. The whole system is then in the state $(D \otimes I_2)(u \otimes b)$.
Now Alice measures her two qubits and tells the result to Bob. The state is projected to a certain subspace depending on the measurement result. If the result is 00, then we project to the subspace $V_{00} := \langle\boxed{0|0|0}, \boxed{0|0|1}\rangle$. If the result is 01, then we project to the subspace $V_{01} := \langle\boxed{0|1|0}, \boxed{0|1|1}\rangle$. If the result is 10, then we project to the subspace $V_{10} := \langle\boxed{1|0|0}, \boxed{1|0|1}\rangle$. If the result is 11, then we project to the subspace $V_{11} := \langle\boxed{1|1|0}, \boxed{1|1|1}\rangle$.
Finally, Bob applies a unitary transformation of the for $I_4 \otimes M_{ij}$, where $M_{ij} \in \mathbb{C}^{2\times 2}$ is one of four fixed unitary matrices $M_{00}, M_{01}, M_{10}, M_{11}$. Which of the four matrices is chosen depends on the measurement result.

Exercise 2: Compute the projection of p_{ij} of $(D \otimes I_2)(u \otimes b)$, for $i, j = 0, 1$. Construct, matrices $M_{00}, M_{01}, M_{10}, M_{11}$, such that the equations

$$(I_4 \otimes M_{00})p_{00} = \boxed{0|0} \otimes u, (I_4 \otimes M_{01})p_{01} = \boxed{0|1} \otimes u,$$

$$(I_4 \otimes M_{10})p_{10} = \boxed{1|0} \otimes u, (I_4 \otimes M_{11})p_{11} = \boxed{1|1} \otimes u$$

hold for arbitrary $u \in \mathbb{C}^2$. Write a short instruction how to do quantum teleportation for the quantum engineer.