Proseminar topis on balancing and quantum mechanics

Your task is to choose a topic, write a short paper with a self-contained presentation, and to present the topic in a talk. Your audience consists of the three lecturers (two of which were not present during the relevant part of the lecture) and the other participants of the proseminar.
The topics can be also extended to a Bachelor thesis.

Topic 1. Continuous rigid bodies. A continuous rigid body is given by a measureable set $Q \in \mathbb{R}^3$ and positive measurable function $\rho : Q \to \mathbb{R}$, the mass density. The total mass is $M = \iiint_Q \rho(x,y,z)dxdydz$, and the center of mass can be computed as $\frac{1}{M}\iiint_Q \rho(x,y,z)(x,y,z)dxdydz$. The moment of inertia is defined as the matrix

$$I := \iiint_Q \rho(x,y,z)I_{xyz}dxdydz,$$

where $I_{xyz} := \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$. Give examples of rigid bodies and their moment of intertia. Which matrices can arise as moments of inertia of a continuous rigid body? Show that for every possible combination of mass, center of mass, and motion of intertia, there is a rectangular hexahedron (Quader) with constant mass density with the same combination.

Topic 2: Balancing a spatial 4R loop. We study the balancing problem for a 4R loop with the following geometric parameters:

1. In link 1, the axes of the joint 4 is $(x,y,z) = (t,0,0)_t$, with assembly points $A_{14} = (0,0,0)$ and $B_{14} = (5,0,0)$; and the axes of joint 1 is $(x,y,z) = (3t,10,4t)_t$, with assembly points $A_{12} = (0,10,0)$ and $B_{12} = (3,10,4)$.
2. In link 2, the axes of the joint 1 is $(x,y,z) = (t,0,0)_t$, with assembly points $A_{21} = (0,0,0)$ and $B_{21} = (5,0,0)$; and the axes of joint 2 is $(x,y,z) = (3t,10,-4t)_t$ with assembly points $A_{23} = (0,10,0)$ and $B_{23} = (3,10,-4)$.
3. In link 3, the axes of the joint 2 is $(x,y,z) = (t,0,0)_t$, with assembly points $A_{32} = (0,0,0)$ and $B_{32} = (5,0,0)$; and the axes of joint 3 is $(x,y,z) = (3t,10,4t)_t$, with assembly points $A_{34} = (0,10,0)$ and $B_{34} = (3,10,4)$.
4. In link 4, the axes of the joint 3 is $(x,y,z) = (t,0,0)_t$, with assembly points $A_{43} = (0,0,0)$ and $B_{43} = (5,0,0)$; and the axes of joint 4 is $(x,y,z) = (3t,10,-4t)_t$, with assembly points $A_{41} = (0,10,0)$ and $B_{41} = (3,10,-4)$.

After redistributing the masses to the axes, we obtain two points on each axes with (maybe negative) points: on joint 2, we get point p_{23} with mass m_{23} and point p_{32} with mass m_{32}, and on joint 3 we get point p_{34} with mass m_{34} and point p_{43} with mass m_{43}. We also get two points on each of the other two joints, but they are not moving when link 1 is the base link, so we ignore them. Note that $m_{32} + m_{34} > 0$ because this sum is the mass of link 3. The 4R loop is perfectly balanced if $p_{23} = p_{32}$ and $p_{34} = p_{43}$ and $m_{23} + m_{32} = 0$ and $m_{34} + m_{43} = 0$.
Prove that it is possible to achieve perfect balance by adding additional masses. Try to minimize the number of links to which such an addition is necessary.

Topic 3: Balancing a planar 4R loop. Assume that the 4 joint axis of a 4R loop are parallel to the z-axis. Then the z-coordinate of all centers of mass never change during motion. As far as the balancing problem is converned, we may project the joint axes to the xy-plane (obtaining points, which are centers of rotation of the xy-plane).
Devise an algorithm for deciding if a 4R loop with 4 parallel joint axes is perfectly balanced. A convenient model is to consider points in the xy-plane as complex numbers. Then the rotation

around a point $j \in \mathbb{C}$ maps $p \in \mathbb{C}$ to $z(p-j)+j$, where $z \in \mathbb{C}$ is a complex number with modulus 1 parametrizing the rotation angle.

Topic 4: The qubit and the Bloch sphere. Let X be a qubit with state space $S_X = \mathbb{C}^2$. An observable of X is defined as a hermitian matrix $A \in \mathbb{C}^{2\times 2}$; here we say that a matrix M is hermitian of $\overline{M}^t = M$. Let $a, b \in \mathbb{R}$ be the eigenvalues of an observable A (it is known that the eigenvalues of an observable are real), and assume $a \neq b$ (if $a = b$, then the measurement result is already known to be a). Let (v_a, v_b) be a basis of eigenvectors of length 1. In order to find the probability distribution for a state $v \in \mathbb{C}$, we write $v = \alpha v_a + \beta v_b$. Then the probabilty for result a is $|\alpha|^2$, and the state after measurement is v_a. Similarily, the probabilty for result b is $|\beta|^2$, and the state after measurement is v_b.
Prove the following facts:

1. For every observable A which is not a multiple of I_2, there are two commensurable observables with value set $\{0, 1\}$ and two commensurable observables with value set $\{+\frac{1}{2}, -\frac{1}{2}\}$. The former are also called Boolean observables, and the latter are called spins. (Hint: commensurability is equivalent to the existence of a common bases of eigenvectors.)

2. Any two spins are conjugated by a (not unique) unitary matrix.

3. There is a bijection between the set of spins and the unit sphere in $\mathbb{R}^3$ such that any conjugation of spins by a unitary matrix corresponds to a rotations of the sphere.

4. Let L_1 and L_2 be spins. Assume that we have just measured L_1 and we got result $+\frac{1}{2}$. Let p_{12} be probability for measuring now $+\frac{1}{2}$. Then p_{12} depends only on the angle between the two unit vectors in $\mathbb{R}^3$ corresponding to L_1 and L_2 (give a formula).

Topic 5. Maximally entangled states. In this topic, we only consider *Boolean* observables on qubits, i.e. the value set is $\{0, 1\}$. All boolean observables can be measured by applying a suitable unitary transformation and then measuring the basic observable.
For a pair of qubits XY, an observable on X is always commensurable with an observable on Y. This has been shown in the lecture for the basic observables; for general observables with value set $\{0, 1\}$, this requires a short explanation.
For a state $v = \alpha\boxed{0} + \beta\boxed{1}$ of XY, we define $\epsilon(v) := |\alpha\delta - \beta\gamma|$ as a measure of entanglement. Prove that the maximal entanglement is $\frac{1}{2}$. Prove that the maximally entangled states have the following property: for any maximally entangled state v, and for any Boolean observable A on X, there is an observable B on Y such that the results of measuring A on X and B on Y is either 01 or 10, (both with probability $\frac{1}{2}$). The following lemmas are suggested:

1. If U is unitary and $v \in S_X \times S_Y$ is a state of XY, then $\epsilon((U \otimes I_2)v) = \epsilon((I_2 1 \otimes U)v) = \epsilon(v)$.

2. If v is maximally entangled, then there exists a unitary matrix U such that $v = (I_2 \otimes U)b$, where $b := \frac{\boxed{0}\boxed{1} - \boxed{1}\boxed{0}}{\sqrt{2}}$ is the Bell state.

3. The Bell state is an eigenvector of $U \otimes U$, for any unitary matrix U (this was shown in the lecture).

Topic 6. Quantum teleportation. As a sort of reverse to superdense coding, Alice may teleport a qubit Z to Bob by communicating two classical bits. Here is the idea: Assume that Alice and Bob share a pair of qubits XY in Bell state, Alice controlling X and Bob controlling Y. Alice applies a certain matrix $A \in \mathbb{C}^{4\times 4}$ to the pair ZX and measures the two basic observables. Then she communicates the results to Bob. Now Bob applies a unitary transformation B_{ij} depending on the bits i, j he got from Alice to Y. Afterwards, the state of Y is equal to the state of Z before the whole process.
Figure out the details: specify the matrices A and B_{ij} for $i, j = 0, 1$, and explain how it works. Hint: Try the matrices D and C_{ij} that were used in superdense encoding.