Journal of Symbolic Computation 87 (2018) 127–139

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Deciding the existence of rational general solutions for first-order algebraic ODEs

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ARTICLE INFO

Article history:
Received 14 October 2016
Accepted 19 June 2017
Available online 27 June 2017

MSC:
34A05
34A26
34A34
14H45
14J26
68W30

Keywords:
Ordinary differential equation
Algebraic curve
Rational parametrization
Rational general solution

ABSTRACT

In this paper, we consider the class of first-order algebraic ordinary differential equations (AODEs), and study their rational general solutions. A rational general solution contains an arbitrary constant. We give a decision algorithm for finding a rational general solution, in which the arbitrary constant appears rationally, of the whole class of first-order AODEs. As a byproduct, this leads to an algorithm for determining a rational general solution of a class of first-order AODE which covers almost all first-order AODEs from Kamke's collection. The method is based intrinsically on the consideration of the AODE from a geometric point of view. In particular, parametrizations of algebraic curves play an important role for a transformation of a parametrizable first-order AODE to a quasi-linear differential equation.

1. Introduction

A first-order algebraic differential equation (AODE) is a differential equation of the form $F(x, y, y') = 0$ for some irreducible trivariate polynomial F with coefficients in an algebraically closed

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[1] The first author was a PhD student at RISC, Johannes Kepler University Linz, while working on this paper and was supported by the strategic program "Innovatives OÖ–2020" by the Upper Austrian Government.

[2] Partially supported by the Austrian Science Fund (FWF): W1214-N15, project DK11.

http://dx.doi.org/10.1016/j.jsc.2017.06.003

field $\mathbb{K}$. First-order AODEs have been studied a lot and there is a variety of solution methods for special classes. The study of such ODEs can be dated back to the work of Fuchs (1884) and Poincaré (1885). Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions in Malmquist (1913), and later Eremenko (1982) revisited this problem. By using the result of Matsuda (1980) on classification of differential algebraic function fields without movable critical points, Eremenko (1998) presented a theoretical consideration on a degree bound of rational solutions.

The problem of finding closed form solutions of first-order AODEs has been considered in several papers. Kovacic (1986) solved completely the problem of computing Liouvilian solutions of a second-order linear ODEs with rational function coefficients. He also proposed an algorithm for determining all rational solutions of a Riccati equation. For the class of first-order first-degree AODEs, Carnicer (1994) studied a degree bound for algebraic solutions in the non-dicritical case. Hubert (1996) found implicit solutions by computing Gröbner bases.

We are mainly interested in rational general solutions, i.e. rational solutions which are also general solutions in the sense of Ritt (1950). Such general solutions must contain a transcendental constant. We take an algebro-geometric approach to this problem. First we neglect the differential aspect and associate to the AODE an algebraic hypersurface. In the case of an autonomous AODE of order one a rational solution of the AODE is a rational parametrization of this hypersurface. In case the hypersurface admits a rational parametrization, we have to look for a reparametrization, which would also satisfy the differential constraint; namely, that the second component of this parametrization should be the derivative of the first one. A similar reasoning is applied in the more complicated situation of non-autonomous AODEs. The algebro-geometric approach has received much attention in the last decade. Algorithms for the class of first-order autonomous AODEs have been proposed in Aroca et al. (2005); Feng and Gao (2004, 2006). The algorithm is based on the fact that if the given AODE has a rational solution, then the algebraic hypersurface obtained from the differential equation by considering the derivative as a new indeterminate is a rational curve. Applying this idea to the general class of first-order AODEs, and combining it with Fuch's theorem on first-order AODEs without movable critical points, Chen and Ma (2005) presented an algorithm for determining a special class of rational general solutions. However, their algorithm is incomplete due to two reasons: the necessary condition for the existence of the solution is not proved to be algorithmically checkable, and a good rational parametrization is required in advance. Ngô and Winkler (2010, 2011b,a) applied the algebro-geometric approach to general non-autonomous first-order AODEs. Using parametrization of algebraic surfaces, they associate to the given parametrizable AODE an associated system of algebraic equations in the parameters. This associated system is a planar rational system. In order to complete the algorithm, a degree bound for irreducible invariant algebraic curves of the planar rational system is required. The problem of finding a uniform bound for the degree of invariant algebraic curves for planar rational systems is known as the Poincaré problem. This difficult problem has been solved by Carnicer (1994), but only generically for the non-dicritical case. So the algorithm of Ngô and Winkler, although producing general rational solutions in almost all situations where such a solution exists, is still no complete decision algorithm.

So far no general algorithm for deciding the existence and, in the positive case, computing a rational general solution of first-order AODEs exists. Such a rational general solution must contain a transcendental constant. If this constant appears rationally, we speak of a strong rational solution. In this paper, we propose a full decision algorithm for taking an arbitrary first-order AODE, deciding the existence of a strong rational general solution, and in the positive case computing such a strong rational general solution. This generalizes the work of Feng and Gao (2004, 2006); Chen and Ma (2005), and Ngô and Winkler (2010, 2011b,a). More specifically, we take an algebro-geometric approach and proceed as follows: we consider the AODE $F(x, y, y') = 0$ as defining a curve over $\mathbb{K}(x)$, the field of rational functions in x over $\mathbb{K}$. I.e., we consider $y' = z$ as a new indeterminate and we view the AODE as an algebraic equation $F(x, y, z) = 0$ defining an algebraic curve in the affine plane $\mathbb{A}^2(\overline{\mathbb{K}(x)})$. We prove that in order for the AODE to have a strong rational general solution, in which the transcendental constant appears rationally, the algebraic curve must be of genus 0. We also prove that over $\mathbb{K}(x)$ every rational curve has an optimal parametrization with coefficients in $\mathbb{K}(x)$, i.e., without algebraically extending the coefficient field. Such an optimal parametrization allows us to transform the given AODE to an associated AODE which is easier to solve. Consequently, we derive a full decision algorithm for deciding the existence of a strong rational general solution; in the positive case, we compute a strong rational general solution.

In Section 2 we present the necessary notations and definitions. In Section 3 the notion of strong solutions is discussed. In Section 4 we prove that rational curves over $\mathbb{K}(x)$ are always parametrizable by rational functions with coefficients in $\mathbb{K}(x)$. The associated equation is derived in Section 5. This leads to the final algorithm for rational general solutions of parametrizable AODEs in Section 6.

2. Preliminaries and notations

By $\mathbb{K}$ we denote an algebraically closed field of characteristic zero. We equip $\mathbb{K}[x]$ with a derivation where $\mathbb{K}$ is the field of constants and $x' = 1$. A first-order algebraic ordinary differential equation (AODE) is a differential equation of the form:

$$F(x, y, y') = 0\,, \tag{1}$$

where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$. W.l.o.g., we assume that F is irreducible. Then $F(x, y, y')$ is irreducible as an element of $\mathbb{K}(x)\{y\}$, the ring of differential polynomials with coefficients in $\mathbb{K}(x)$. By Ritt (1950), the radical differential ideal generated by F, say $\{F\}$, can be decomposed as

$$\{F\} = \left(\{F\} : \frac{\partial F}{\partial y'}\right) \cap \left\{F, \frac{\partial F}{\partial y'}\right\}.$$

The first component on the right hand side, the quotient of $\{F\}$ by the separant of F, is a prime differential ideal. Thus, it has a generic zero. The following definition is due to Ritt (1950).

Definition 2.1. A generic zero of the differential prime ideal $\left(\{F\} : \frac{\partial F}{\partial y'}\right)$ is called a *general solution* of the differential equation (1).

The following lemma is useful for checking whether a solution is general; cf. Ritt (1950, Chap. II.5 and 6).

Lemma 2.2. *A solution y(x) of the differential equation (1) is a general solution if and only if*

$$\forall H \in \mathbb{K}(x)\{y\}: \; H(y(x)) = 0 \Longrightarrow \operatorname{prem}(H, F) = 0\,,$$

where prem *is the differential pseudo remainder.*

Definition 2.3. A general solution $y(x)$ of the differential equation (1) is called a *rational general solution* if it has the form

$$y(x) = \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_m x^m}$$

for some $m, n \in \mathbb{N}$ and a_i, b_j constants in a differential field extension of $\mathbb{K}$.

Our approach towards solving first-order AODEs is an algebro-geometric one. Consider a first-order AODE, $F(x, y, y') = 0$, for an irreducible polynomial F. We first neglect the differential aspect of the equation by replacing the derivative by an independent variable, thus arriving at the algebraic equation $F(x, y, z) = 0$. By considering F in $\mathbb{K}(x)[y, z]$, we let it define a curve in the affine plane over the algebraic closure of the field of rational functions $\mathbb{K}(x)$.

Definition 2.4. The algebraic curve $\mathcal{C}_F$ in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $F(x, y, z) = 0$ is called the *corresponding curve* of the differential equation $F(x, y, y') = 0$.

Typically, given a field of characteristic zero $\mathbb{F}$, we consider a polynomial $G \in \mathbb{F}[y, z]$ as implicitly defining an algebraic curve in the affine plane over the algebraic closure of $\mathbb{F}$, $\mathbb{A}^2(\overline{\mathbb{F}})$, by the equation $G(y, z) = 0$. We say that this curve is defined over $\mathbb{F}$. Often it is useful to have a parametric representation of the points on the curve.

Definition 2.5. Let $\mathcal{C} \subset \mathbb{A}^2(\overline{\mathbb{F}})$ be an algebraic curve defined over $\mathbb{F}$.

A *rational parametrization*, or briefly, a *parametrization* of $\mathcal{C}$ is a rational map $\mathcal{P} : \mathbb{A}^1(\overline{\mathbb{F}}) \to \mathcal{C}$ such that the image of $\mathcal{P}$ is dense in $\mathcal{C}$ (w.r.t. the Zariski topology).

If, furthermore, $\mathcal{P}$ is a birational equivalence, $\mathcal{P}$ is called a *proper* parametrization.

If $\mathcal{C}$ admits a parametrization, we say $\mathcal{C}$ is a *rational curve*.

Only irreducible curves can have a rational parametrization. Some of the results in this paper work also for the more general radical parametrizations. For details we refer to Sendra and Sevilla (2011). In this paper we stick to the rational case.

It is well-known that if an algebraic curve admits a rational parametrization, then it admits a proper parametrization. The following theorem gives a necessary and sufficient condition for an algebraic curve to have a rational parametrization.

Theorem 2.6 *(Rationality criterion). An algebraic curve is rational if and only if its genus is equal to zero.*

In case the curve $\mathcal{C}$ is rational, it has infinitely many proper parametrizations. We are interested in the extension field of $\mathbb{F}$ of least extension degree, which contains the coefficients of a parametrization of $\mathcal{C}$.

Definition 2.7. Let $\mathcal{P}(t) = (p_1(t), p_2(t)) \in \overline{\mathbb{F}}(t)^2$ be a parametrization of the curve $\mathcal{C}$. Let $\mathbb{L}$ be the extension field of $\mathbb{F}$ by the coefficients in $\mathcal{P}$. We call $\mathbb{L}$ the *field of coefficients* of $\mathcal{P}$. We say that $\mathcal{C}$ is *parametrizable over* a field K, if K contains the field of coefficients of a parametrization of $\mathcal{C}$.

Definition 2.8. Let $\mathbb{L}$ be an algebraic extension of $\mathbb{F}$. A point $(y_0, z_0) \in \mathcal{C}$ is called an $\mathbb{L}$-rational point iff $(y_0, z_0) \in \mathbb{A}^2(\mathbb{L})$.

The following proposition is a criterion for a rational curve to be parametrizable over a given field; see for instance Sendra et al. (2008).

Proposition 2.9. *Given an algebraic extension field $\mathbb{L}$ of $\mathbb{F}$, the rational curve $\mathcal{C}$ is parametrizable over $\mathbb{L}$ if and only if $\mathcal{C}$ contains a simple $\mathbb{L}$-rational point.*

A parametrization whose field of coefficients is as small as possible is called an optimal parametrization. Several algorithms for determining an optimal parametrization of a rational curve have been given. For further details on parametrization of algebraic curves we refer to Sendra et al. (2008).

3. Strong rational general solutions

In this section, we give a necessary condition for a first-order AODE to admit a rational general solution in which the transcendental constant c appears rationally; i.e., which is of the form $y(x, c) \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$. The following theorem is a slightly different version of Theorem 2.4 in Chen and Ma (2005). We carefully distinguish rational general solutions in the sense of Ritt (1950) and those used in Chen and Ma (2005). The latter ones will be called strong rational general solutions. Together with new investigations about parametrization of algebraic curves over rational function fields in the next section, the discussion in this section shows how to extend the method of Chen and Ma (2005) to a full algorithm. Note, that we assume irreducibility in $\mathbb{K}[x, y, z]$.

Theorem 3.1. *Let F be an irreducible polynomial in $\mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$. If the differential equation $F(x, y, y') = 0$ has a rational solution of the form $y(x, c) \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$ for an arbitrary constant c, then its corresponding curve $\mathcal{C}_F$ in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ is rational, and admits a parametrization with coefficients in $\mathbb{K}(x)$.*

Proof. First, we need to prove that F is still irreducible as a polynomial in $\overline{\mathbb{K}(x)}[y, z]$. In order to do that, let us consider the ideal

$$I := \{H \in \overline{\mathbb{K}(x)}[y, z] \mid H(x, y(x, c), y'(x, c)) = 0\}$$

in the polynomial ring $\overline{\mathbb{K}(x)}[y, z]$. We claim that I is a principle prime ideal. Consider the ring homomorphism $\phi : \overline{\mathbb{K}(x)}[y, z] \to \overline{\mathbb{K}(x)}(c)$, defined by $\phi(H) := H(x, y(x, c), y'(x, c))$ for $H \in \overline{\mathbb{K}(x)}[y, z]$. The kernel of ϕ is exactly I. Therefore ϕ induces an embedding from the quotient ring $\overline{\mathbb{K}(x)}[y, z]/I$ to $\overline{\mathbb{K}(x)}(c)$. Thus $\overline{\mathbb{K}(x)}[y, z]/I$ is a domain, and then I is a prime ideal. Since $\overline{\mathbb{K}(x)}[y, z]$ is a noetherian unique factorization domain, we know from Hartshorne (1977, Prop. 1.12A, p. 7) that every prime ideal of height one is principle. Hence, I is principle.

Next we prove that I can be generated by an irreducible polynomial G in $\mathbb{K}[x, y, z]$. We construct such a generator by the method of Gröbner bases. Let $y(x, c) = \frac{P_1(x,c)}{P_2(x,c)}$ and $y'(x, c) = \frac{Q_1(x,c)}{Q_2(x,c)}$ be in reduced form, i.e. $P_1, P_2, Q_1, Q_2 \in \mathbb{K}[x, c]$ such that $\gcd(P_1, P_2) = \gcd(Q_1, Q_2) = 1$. From the definition of the ideal I, we know by implizitation that

$$I = \langle yP_2 - P_1, \, zQ_2 - Q_1, 1 - P_2t_1, 1 - Q_2t_2\rangle \cap \overline{\mathbb{K}(x)}[y, z] \, .$$

The first component on the right hand side is an ideal in $\overline{\mathbb{K}(x)}[c, t_1, t_2, y, z]$ generated by the polynomials $yP_2 - P_1, zQ_2 - Q_1, 1 - P_2t_2$ and $1 - Q_2t_2$. We fix the lexicographic ordering on $\overline{\mathbb{K}(x)}[c, t_1, t_2, y, z]$ with $c > t_1 > t_2 > y > z$. Using this ordering we compute a reduced Gröbner basis of I by first computing a reduced Gröbner basis for the first component of the right hand side, and then eliminating all elements containing c, t_1, t_2. Buchberger's algorithm and reduction of the obtained basis yields a list of polynomials in the variables c, t_1, t_2, y, z with coefficients in $\mathbb{K}(x)$. Therefore, after eliminating polynomials containing c, t_1, t_2, we obtain a reduced Gröbner basis of I which contains only polynomials in $\mathbb{K}(x)[y, z]$. Since I is principle, the reduced Gröbner basis of I contains only one element, say $G_1 \in \mathbb{K}(x)[y, z]$. Moreover, since I is a prime ideal, G_1 must be irreducible over $\overline{\mathbb{K}(x)}[y, z]$ and hence also in $\mathbb{K}(x)[y, z]$. Let $G \in \mathbb{K}[x, y, z]$ such that $G_1 = \frac{a(x)}{b(x)}G$ for some $a(x), b(x) \in \mathbb{K}[x]$ and G is primitive over $\mathbb{K}[x]$. Hence, G is irreducible over $\mathbb{K}(x)[y, z]$ (since G_1 is irreducible). Then we have $I = \langle G_1\rangle = \langle G\rangle$ over $\overline{\mathbb{K}(x)}[y, z]$. Therefore, G is irreducible over $\overline{\mathbb{K}(x)}[y, z]$.

Since F is an irreducible element in the ideal I, F differs from G only by a non-zero constant factor in $\mathbb{K}$. Therefore, F is also irreducible over $\overline{\mathbb{K}(x)}[y, z]$, and consequently the corresponding curve $\mathcal{C}_F$ is irreducible. Since $F(x, y(x, c), y'(x, c)) = 0$, the curve $\mathcal{C}_F$ can be parametrized by a pair of rational functions $\mathcal{P}(t) := (y(x, t), \frac{\partial}{\partial x} y(x, t))$. Hence $\mathcal{C}_F$ is rational and admits a parametrization with coefficients in $\mathbb{K}(x)$. □

Theorem 3.1 motivates the following definitions.

Definition 3.2. The first-order AODE $F(x, y, y') = 0$ is called *parametrizable* if its corresponding curve is rational.

All differential equations of the form $y'F_1(x, y) = F_0(x, y)$, where $F_0, F_1 \in \mathbb{K}[x, y]$, are parametrizable. As a consequence, we might also say that all quasi-linear differential equations of the form $y' = \frac{F_0(x,y)}{F_1(x,y)}$ are parametrizable.

Note, that 89 percent of the first-order AODEs listed in the collection of Kamke (1983) are parametrizable. The remaining ones consist of two classes. One part contains the reducible AODEs, hence, parametrizability of the factors can be considered. Around one half of the reducible AODEs have parametrizable factors. The other part consists of AODEs for which the corresponding curve has genus greater than 0.

The class of first-order AODEs covers around 64 percent of the entire collection of first-order ODEs in Kamke. Some of the remaining first-order ODEs contain arbitrary functions. For certain choices of these functions, the ODEs might be algebraic. For further details on statistical investigations of Kamke's list we refer to Grasegger et al. (2015).

The classical definition of a general solution is that of a solution containing some arbitrary constant. In fact it is easy to show that a rational general solution in the sense of Ritt (Definition 2.1) needs to contain a coefficient, which is not in $\mathbb{K}$. Assume to the contrary that $\bar{y}(x) = \frac{P(x)}{Q(x)}$, with $P, Q \in \mathbb{K}[x]$ is a rational general solution. Then $\bar{y}$ is a zero of the polynomial $Q(x)y - P(x)$. The differential pseudo remainder of this polynomial with F is the polynomial itself, which is a contradiction to Lemma 2.2.

A rational general solution of the differential equation (1) is not necessarily of the form $y(x, c) \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$ for some transcendental constant c. However, if $y(x, c)$ is a solution of the differential equation (1), then it is a general solution in the sense of Ritt. In fact, let us assume that $H \in K(x)\{y\}$ is an arbitrary differential polynomial such that $H(y(x, c)) = 0$, and that $G := \mathrm{prem}(H, F)$. Then $G \in \mathbb{K}(x)[y, y']$. From the definition of pseudo differential remainder, we know that there are natural numbers m, n such that $S_F^m I_F^n G - H$ is a linear combination of F and its derivatives with coefficients in $\mathbb{K}(x)\{y\}$, where S_F and I_F are separant and initial of F, respectively. S_F and I_F do not vanish at $y = y(x, c)$. Otherwise, as we have seen in the proof of Theorem 3.1, S_F and I_F differ from F only by a factor in $\mathbb{K}(x)$, which is not possible. Therefore G vanishes at $y = y(x, c)$. So G differs from F only by a factor in $K(x)$. This implies $G = 0$. Finally, Lemma 2.2 asserts that $y(x, c)$ is a general solution.

Definition 3.3. A solution y of the differential equation (1) is called a *strong rational general solution* if $y = y(x, c) \in \mathbb{K}(x, c) \setminus \mathbb{K}(x)$, where c is a transcendental constant over $\mathbb{K}(x)$.

Theorem 3.1 is not true if the given rational general solution is not strong. For instance, the differential equation

$$x^3 y'^3 - (3x^2 y - 1)y'^2 + 3xy^2 y' - y^3 + 1 = 0$$

has a rational general solution

$$y(x) = cx + (c^2 + 1)^{\frac{1}{3}},$$

which is not strong. The corresponding curve has genus 1. Therefore, the differential equation has no strong rational general solution. However, as we will see later, if a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.

4. Optimal parametrizations of rational curves over $\mathbb{K}(x)$

We have seen that the corresponding curve of a first-order AODE having a strong rational general solution is rational. Moreover, by Theorem 3.1 the corresponding curve admits a parametrization with coefficients in $\mathbb{K}(x)$. In case we have a parametrization with coefficients in $\mathbb{K}(x)$ we can decide the existence of a strong rational general solution and compute it. Indeed, as we show in this section, such a parametrization always exists.

Optimal parametrization is a key notion for answering the question. Several algorithms for determining an optimal parametrization of a rational curve are available. Sendra et al. (2008) proposed an algorithm for computing an optimal parametrization of a rational curve over the field $\mathbb{Q}$ of rational numbers. A similar result for the class of rational curves over the field $\mathbb{Q}(x)$ of rational functions is presented in Hillgarter and Winkler (1997). By different methods, Beck and Schicho (2008) studied the optimal parametrization problem for rational curves over perfect fields. Since $\mathbb{K}(x)$ is a perfect field, the algorithm of Beck and Schicho is applicable over $\mathbb{K}(x)$. Below, we follow the idea by Hillgarter and Winkler (1997) to determine an optimal parametrization of a rational curve over $\mathbb{K}(x)$.

Let us fix a rational curve $\mathcal{C}$ in $\mathbb{A}^2(\overline{\mathbb{K}(x)})$ defined by $G(x, y, z) = 0$, for some irreducible polynomial $G \in \mathbb{K}(x)[y, z]$. As a consequence of a theorem by Hilbert and Hurwitz (Sendra et al., 2008, Ch. 5, p. 152), $\mathcal{C}$ can be birationally transformed to a line or a conic over $\mathbb{K}(x)$, depending on whether the total degree of G is odd or even, respectively. The transformation is described in Sendra et al. (2008)

by using the notion of adjoint curves. A line is always parametrizable over $\mathbb{K}(x)$. To parametrize a conic over $\mathbb{K}(x)$, it suffices to find a $\mathbb{K}(x)$-rational point on it.

In the following we show, along the lines of Hillgarter and Winkler (1997), that indeed there always exists such a $\mathbb{K}(x)$-rational point. Conics are classified in the usual way as parabolas, hyperbolas and ellipses. In case the conic is a parabola, a $\mathbb{K}(x)$-rational point can always be constructed by a formula similar to the one in Hillgarter and Winkler (1997, p. 195) for the field $\mathbb{Q}$. Note that every conic over $\mathbb{K}(x)$ can be linearly transformed to a projective conic in $\mathbb{P}^2(\mathbb{K}(x))$ of the form $AY^2 + BZ^2 - W^2 = 0$ for some square-free polynomials $A, B \in \mathbb{K}[x]$ (see Vo, 2016, Chp. 4, Sec. 4.2).

Proposition 4.1. *For any square-free polynomials $A, B \in \mathbb{K}[x]$, the projective conic defined by $AY^2 + BZ^2 - W^2 = 0$ always has a $\mathbb{K}(x)$-rational point.*

Before giving a proof for this proposition, we need the following lemma.

Lemma 4.2. *Let A, B be polynomials in $\mathbb{K}[x]$ such that A is square-free and $\deg A \geq \deg B \geq 1$. Then there exist $a, b, m \in \mathbb{K}[x]$ such that a is square-free, $\deg a < \deg A$, and $b^2 - B = am^2 A$.*

Proof. Denote by n the degree of A and let $x_1, \ldots, x_n \in \mathbb{K}$ be the roots of A. There exists a polynomial $b \in \mathbb{K}[x]$ of degree at most $n-1$ such that $b(x_i) = \sqrt{B(x_i)}$ for every $i = 1, \ldots, n$, where $\sqrt{B(x_i)}$ is a square root of $B(x_i)$. We see that $B(x) \equiv b(x)^2 \mod (x - x_i)$ for every $i = 1, \ldots, n$. Since A is square-free, we have $B(x) \equiv b(x)^2 \mod A(x)$.

Now let $a, m \in \mathbb{K}[x]$ such that a is square-free and $\frac{b^2 - B}{A} = a \cdot m^2$. Note that such a pair (a, m) is always exist. It remains to prove that $\deg a < \deg A$. Indeed, we have

$$\begin{aligned} \deg a &= \deg(b^2 - B) - \deg(Am^2) \\ &\leq \deg(b^2 - B) - \deg A \\ &\leq \max\{2(\deg A - 1), \deg B\} - \deg A \\ &< \deg A\,. \qquad \square \end{aligned}$$

From the proof, we see that $\deg b \leq \deg A - 1$. This fact leads us to an algorithmic way to determine the triple (a, b, m) by the method of indeterminate coefficients. In particular, we first set b a polynomial of degree $\deg A - 1$ in x with indeterminate coefficients. Since A divides $b^2 - B$, the remainder must be equal to zero. This yields an algebraic system in the indeterminate coefficients. By solving this algebraic system, we can find all possible choices for b, and hence for a and m.

Proof of Proposition 4.1. This proof follows the lines of Hillgarter and Winkler (1997).

Let $A, B \in \mathbb{K}[x]$ be square-free polynomials, and consider the projective conic $\mathcal{E}$ defined by $AY^2 + BZ^2 - W^2 = 0$. Denote $d(\mathcal{E}) := \min(\deg A, \deg B)$. We prove the existence of a $\mathbb{K}(x)$-rational point on $\mathcal{E}$ by induction on $d(\mathcal{E})$. In the induction base case, i.e. $d(\mathcal{E}) = 0$, for instance $\deg A = 0$, we see that $(1 : 0 : \sqrt{A}) \in \mathbb{P}^2(\mathbb{K}(x))$ is a $\mathbb{K}(x)$-rational point of the conic.

Let $m \geq 1$ be an arbitrary natural number, and assume that for every projective conic $\tilde{\mathcal{E}}$ defined by $\tilde{A}Y^2 + \tilde{B}Z^2 - W^2 = 0$ for some square-free polynomials $\tilde{A}, \tilde{B} \in \mathbb{K}[x]$, if $d(\tilde{\mathcal{E}}) < m$ then $\tilde{\mathcal{E}}$ admits a $\mathbb{K}(x)$-rational point. We need to prove that if $d(\mathcal{E}) = m$, then $\mathcal{E}$ also admits a $\mathbb{K}(x)$-rational point.

In case $d(\mathcal{E}) = m$, we proceed as follows. We may assume that $\deg A \geq \deg B = m$, otherwise we just swap Y and Z. By Lemma 4.2, there exist $A_1, b, m \in \mathbb{K}[x]$ such that A_1 is square-free, $\deg A_1 < \deg A$, and $b^2 - B = A_1 m^2 A$. We transform the coordinate system (Y, Z, W) to the new one $(\overline{Y}, \overline{Z}, \overline{W})$ by the linear transformation

$$\begin{bmatrix} \overline{Y} \\ \overline{Z} \\ \overline{W} \end{bmatrix} = \begin{bmatrix} Am & 0 & 0 \\ 0 & b & 1 \\ 0 & B & b \end{bmatrix} \begin{bmatrix} Y \\ Z \\ W \end{bmatrix}.$$

Algorithm 1 OptimalParametrization (Optimal Parametrization)

Require: A rational curve $\mathcal{C}$ defined over $\mathbb{K}(x)$
Ensure: An optimal proper parametrization for $\mathcal{C}$

1: Determine a birational transformation, say $\mathcal{T}$, to transform the curve $\mathcal{C}$ to a birationally equivalent curve $\mathcal{E}$, which is either a line or a conic. This is achieved by an algorithm derived from the proof of the theorem of Hilbert and Hurwitz (see Theorem 5.8 and Algorithm Hilbert–Hurwitz in Sendra et al., 2008).
2: **if** $\mathcal{E}$ is a line **then**
3: Determine an optimal parametrization $\mathcal{P}(t)$ for the line.
4: **else if** $\mathcal{E}$ is a conic **then**
5: Linearly transform the conic $\mathcal{E}$ to a projective conic $\mathcal{E}'$ of the form $AY^2 + BZ^2 - W^2 = 0$ for some square-free polynomials $A, B \in \mathbb{K}[x]$.
6: Construct a $\mathbb{K}(x)$-rational point Q' on $\mathcal{E}'$ by the method described in the proof of Proposition 4.1.
7: Determine the corresponding $\mathbb{K}(x)$-rational point Q on $\mathcal{E}$.
8: Determine a proper parametrization $\mathcal{P}(t)$ for $\mathcal{E}$ by using the point Q. See Algorithm Conic-Parametrization in Sendra et al. (2008, p. 115).
9: **end if**
10: **return** $\mathcal{T}^{-1}(\mathcal{P}(t))$

Then we see that

$$A_1\overline{Y}^2 + B\overline{Z}^2 - \overline{W}^2 = (b^2 - B)(AY^2 + BZ^2 - W^2)\,.$$

Since B is square-free, $b^2 - B \neq 0$. Thus the conic $\mathcal{E}$ has a $\mathbb{K}(x)$-rational point if and only if the projective conic $\mathcal{E}_1$ defined by $A_1\overline{Y}^2 + B\overline{Z}^2 - \overline{W}^2 = 0$ has a $\mathbb{K}(x)$-rational point.

If $\deg A_1 < \deg B$, then $d(\mathcal{E}_1) = \deg A_1 < \deg B = m$. Therefore $\mathcal{E}_1$ satisfies the induction hypothesis. So $\mathcal{E}_1$ admits a $\mathbb{K}(x)$-rational point, and consequently so does $\mathcal{E}$.

In case $\deg A_1 \geq \deg B$, we can repeat the above process recursively until we get a projective conic $\mathcal{E}_k$ defined by $A_kY^2 + BZ^2 - W^2 = 0$, where A_k is square-free and $\deg A_k < \deg B$. Note, that the polynomial B remains unchanged by these transformations. At this point we have $d(\mathcal{E}_k) = \deg A_k < \deg B = m$. Therefore $\mathcal{E}_k$ satisfies the induction hypothesis. So $\mathcal{E}_k$ admits a $\mathbb{K}(x)$-rational point, and consequently so does $\mathcal{E}$. □

The proof is constructive. We now conclude the above discussion by the following theorem.

Theorem 4.3. *A rational curve defined over $\mathbb{K}(x)$, i.e. a curve which can be parametrized over $\overline{\mathbb{K}(x)}$, can actually be parametrized over $\mathbb{K}(x)$. So optimal parametrizations of a rational curve over $\mathbb{K}(x)$ always have coefficients in $\mathbb{K}(x)$.*

Furthermore, an algorithm for determining such an optimal parametrization can be described in analogy to the process of Hillgarter and Winkler (1997). Algorithm 1 (OptimalParametrization) summarizes the above discussion on computing an optimal parametrization over $\mathbb{K}$.

5. Associated differential equations

In this section, we only work with the class of parametrizable first-order AODEs. Based on optimal parametrizations of the corresponding curves, we construct for each parametrizable first-order AODE an associated differential equation, which is a quasi-linear ordinary differential equation. Several facts about connections between rational general solutions of a parametrizable first-order AODE and its associated differential equation will be presented. The problem which remains is looking for rational general solutions of quasi-linear differential equations. This problem is discussed at the end of this section.

Consider a parametrizable first-order AODE (1), $F(x, y, y') = 0$, and assume that an optimal parametrization $\mathcal{P} = (p_1, p_2) \in (\mathbb{K}(x)(t))^2$ of the corresponding curve is given, where we write $p_i(t) = p_i(x, t)$ to indicate the dependence on x. Let $y(x) \in \overline{\mathbb{K}(x)}$ be an algebraic solution. Then the pair of two algebraic functions $(y(x), y'(x))$ can be seen as an algebraic solution point on the corresponding curve $\mathcal{C}$. Two cases arise.

(i) $(y(x), y'(x)) \notin \mathrm{im}(\mathcal{P})$, where $\mathrm{im}(\mathcal{P})$ is the image of $\mathcal{P}$. Then $(y(x), y'(x))$ is contained in the finite set $\mathcal{C} \setminus \mathrm{im}(\mathcal{P})$.
(ii) $(y(x), y'(x)) = \mathcal{P}(\omega(x))$ for some $\omega(x) \in \overline{\mathbb{K}(x)}$. In this case we identify the algebraic function $\omega(x)$ with a point on the affine line $\mathbb{A}^1(\overline{\mathbb{K}(x)})$.

Let us take a look at the algebraic function $\omega(x)$. It satisfies the system

$$\begin{cases} p_1(x, \omega(x)) = y(x)\,, \\ p_2(x, \omega(x)) = y'(x)\,. \end{cases}$$

Therefore,

$$\frac{d}{dx} p_1(x, \omega(x)) = p_2(x, \omega(x))\,.$$

By expanding the left hand side, we get

$$\omega'(x) \cdot \frac{\partial p_1}{\partial t}(x, \omega(x)) + \frac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))\,.$$

Thus $\omega(x)$ either satisfies the algebraic relations

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(x, \omega(x)) = 0\,, \\ \dfrac{\partial p_1}{\partial x}(x, \omega(x)) = p_2(x, \omega(x))\,, \end{cases}$$

or it is an algebraic solution of the quasi-linear differential equation

$$\omega' = \frac{p_2(x, \omega) - \frac{\partial p_1}{\partial x}(x, \omega)}{\frac{\partial p_1}{\partial t}(x, \omega)}\,. \tag{2}$$

The ODE (2) will be of further importance. Note, that this ODE has been already discussed in Fuchs (1884) and the idea was also used in Chen and Ma (2005). Since it is an essential part of the method we nevertheless elaborate details in our notation and setting.

Definition 5.1. Let $F(x, y, y') = 0$ be a first-order AODE and consider a proper rational parametrization $\mathcal{P}(t) = (p_1(x, t), p_2(x, t))$ of the corresponding curve $\mathcal{C}_F$. Then the ODE (2) is called the *associated differential equation* to F.

By the reasoning above, we have proven the following lemma.

Lemma 5.2. *With notations as above, if $y = y(x) \in \overline{\mathbb{K}(x)}$ is an algebraic solution of the differential equation (1), then one of the following holds:*

(i) *The algebraic solution point $(y(x), y'(x))$ is contained in the finite set $\mathcal{C} \setminus \mathrm{im}(\mathcal{P})$.*
(ii) *$y(x) = p_1(x, \omega(x))$ for some algebraic solution $\omega(x)$ of the algebraic system*

$$\begin{cases} \dfrac{\partial p_1}{\partial t}(x, \omega) = 0\,, \\ \dfrac{\partial p_1}{\partial x}(x, \omega) = p_2(x, \omega)\,. \end{cases}$$

(iii) *$y(x) = p_1(x, \omega(x))$ for some algebraic solution $\omega(x)$ of the associated differential equation (2).*

Note that a rational general solution can be seen as a one-parameter family of rational solutions. Therefore, in the above lemma, if $y(x)$ is a rational general solution, then (i) and (ii) cannot happen. So we focus on (iii).

Theorem 5.3. *We use the notation from above and assume that the parametrization $\mathcal{P}$ is proper. Then there is a one-to-one correspondence between rational general solutions of the differential equation (1) and rational general solutions of its associated differential equation (2).*

In particular, if $\omega(x)$ is a rational general solution of the associated equation (2), then $y(x) = p_1(x, \omega(x))$ is a rational general solution of (1).

Conversely, if $y(x)$ is a rational general solution of (1), then $\omega(x) = \mathcal{P}^{-1}(y(x), y'(x))$ is a rational general solution of the associated equation (2).

Proof. Assume that $\omega(x)$ is a rational general solution of the associated differential equation (2), and denote $y(x) := p_1(x, \omega(x))$. From the construction above, it is clear that $y(x)$ is a rational solution of the differential equation (1).

It remains to show that $y(x)$ is a general solution. Let $G \in \mathbb{K}(x)\{y\}$ be a differential polynomial such that $G(y(x)) = 0$, and let $H := \mathrm{prem}(G, F)$. We need to show that $H = 0$. Since y' is the highest derivative occurring in F, we know that $H \in \mathbb{K}(x)[y, y']$. Both G and F vanish at $y(x)$, hence so does H regarded as a differential polynomial. Therefore, $H(\mathcal{P}(\omega(x))) = H(y(x), y'(x)) = 0$ regarding H as a polynomial. Note, that $(H \circ \mathcal{P})(\omega) = H(f_1(x, \omega), f_2(x, \omega)) \in \mathbb{K}(x, \omega)$. Since $\omega(x)$ is a general solution of the associated differential equation, it contains an arbitrary constant. Thus $\omega(x)$ cannot be a root of a non-zero rational function in $\mathbb{K}(x, \omega)$. In particular, since $(H \circ \mathcal{P})(\omega(x)) = 0$, we obtain $H \circ \mathcal{P} = 0$. Then H vanishes on $\mathcal{C}_F$. This implies that F divides H. But H is a pseudo remainder (of G) with respect to F, so $H = 0$.

On the other hand, if $y(x)$ is a rational general solution of (1), then, by the construction of the associated equation, $\omega(x) := \mathcal{P}^{-1}(y(x), y'(x))$ is a rational solution of (2). By a similar argument as above ω is a rational general solution of the associated differential equation (2). □

Lemma 5.2 tells us that for finding rational solutions of a parametrizable first-order AODE, working with the class of quasi-linear first-order ODEs is essentially enough. If we look for rational general solutions, the situation is even much stricter. Fuchs (1884) gave a necessary and sufficient condition for a first-order AODE to have no movable branch point (see also Ince, 1956, Chp. 13). Note that if a first-order AODE admits a rational general solution, it has no movable branch point. Thus, as an application of Fuchs' theorem, a quasi-linear first-order AODE has a rational general solution only if it is a linear differential equation or a Riccati equation. By a different approach, Behloul and Cheng (2011) also proved that a quasi-linear first-order AODE which is neither linear nor Riccati can have at most finitely many rational solutions. The following theorem is a combination of Theorem 5.3 and the above discussion.

Theorem 5.4. *Let $F(x, y, y') = 0$ be a first-order AODE.*

(i) *If $F = 0$ has a strong rational general solution, then it is parametrizable and its associated differential equation is of the form*

$$\omega' = a_0(x) + a_1(x)\omega + a_2(x)\omega^2 \,, \tag{3}$$

for some $a_0, a_1, a_2 \in \mathbb{K}(x)$.

(ii) *If $F = 0$ is parametrizable and has a rational general solution, then its associated quasi-linear differential equation is of the form (3).*

Proof. If a parametrizable first-order AODE has a rational general solution, then so does its associated differential equation. In this case the associated differential equation has infinitely many rational solutions. Then (ii) follows from the results of Fuchs (1884) or Behloul and Cheng (2011). Finally, (i) follows immediately from Theorem 3.1 and (ii). □

In the requirement of the results by Fuchs (1884) and Behloul and Cheng (2011), the coefficients of the quasi-linear differential equation must be rational functions. Algorithm 1 always produces an optimal parametrization with rational function coefficients (if there is any). This guarantees that the associated differential equations meet the requirement.

Algorithm 2 SRGS (Strong rational general solutions of first-order AODEs)

Require: A first-order AODE, $F(x, y, y') = 0$, where $F \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, y]$ is irreducible.
Ensure: A strong rational general solution $y(x)$, or “No strong rational general solution exists”.

if F is irreducible over $\overline{\mathbb{K}(x)}$ and the genus of the corresponding curve is zero **then**
  Use Algorithm 1 OPTIMALPARAMETRIZATION to compute an optimal proper parametrization of the corresponding curve, say $(p_1(x, t), p_2(x, t)) \in (\mathbb{K}(x)(t))^2$.
  Compute

$$f(x, t) := \frac{p_2(x, t) - \frac{\partial}{\partial x} p_1(x, t)}{\frac{\partial}{\partial t} p_1(x, t)}.$$

  if $f(x, t)$ has the form $a_0(x) + a_1(x)t + a_2(x)t^2$ for some $a_0, a_1, a_2 \in \mathbb{K}(x)$ **then**
    consider the linear or Riccati equation $\omega' = f(x, \omega)$
    if it has a rational general solution, say $\omega(x)$ **then**
      return $y(x) = p_1(x, \omega(x))$
    end if
  end if
end if
return “No strong rational general solution exists”.

Corollary 5.5. *If a parametrizable first-order AODE has a rational general solution, then it has a strong rational general solution.*

Proof. This is a consequence of the previous theorem and Schwarz (2008, Cor. 2.1, p. 18). □

Remark 5.6. Ngô and Winkler (2010) proved that if a first-order AODE is parametrizable as an algebraic surface over $\mathbb{K}$, and that if it has a rational general solution, then it has a strong rational general solution. Due to Theorem 3.1, the AODE must be parametrizable as an algebraic curve over $\mathbb{K}(x)$. Note that a parametrizable first-order AODE is always parametrizable as an algebraic surface over $\mathbb{K}$. Thus, if a first-order AODE admits a rational general solution, then it is parametrizable as an algebraic surface over $\mathbb{K}$ if and only if it is parametrizable as an algebraic curve over $\mathbb{K}(x)$.

We are looking for rational general solutions of first-order AODEs. The problem has been reduced to computing a rational general solution of the differential equation (3). In the case $a_2 = 0$, (3) is a linear differential equation of degree 1 which can be easily solved by integration. In the case $a_2 \neq 0$, it is a classical Riccati equation.

For the problem of computing a rational general solution, or even all rational solutions, of a Riccati equation, we refer the reader to Kovacic (1986) for a complete algorithm. Kovacic (1986) proposes an algorithm for computing Liouvilian solutions of a linear second-order AODE. As a special case, Section 3.1 in that paper leads to a full algorithm for determining all rational solutions of a Riccati equation. Note that for a Riccati equation, the notion of rational general solutions and strong rational general solutions coincide. Chen and Ma (2005) slightly modify the algorithm by Kovacic to look only for strong rational general solution.

6. The decision algorithm

This section is devoted to an algorithm for finding strong rational general solutions of first-order AODEs. As we have seen before, if a first-order AODE has a strong rational general solution, then it is parametrizable, i.e. its corresponding curve is rational. Whenever a first-order AODE is parametrizable, the notions of rational general solution and strong rational general solution coincide. Moreover, in the case of having a strong rational general solution, the associated ODE is either a linear differential equation or a Riccati equation.

In Algorithm 2 SRGS we present a full algorithm which computes for a given first-order AODE a strong rational general solution, if it exists. Otherwise it decides that such a solution cannot exist.

Theorem 6.1. *Algorithm 2* SRGS *returns a strong rational general solution of the given first-order AODE, $F(x, y, y') = 0$, if there is any; and it returns "No strong rational general solution exists" if the differential equation has no strong rational general solution.*

Hence, Algorithm 2 SRGS decides the existence of strong rational general solutions of the whole class of first-order AODEs. Furthermore, due to Corollary 5.5, Algorithm 2 SRGS can also be used for determining the existence of rational general solutions of parametrizable first-order AODEs. In the affirmative case it always computes such a solution.

Example 6.2 *(Example 1.537 in Kamke, 1983).* Consider the differential equation

$$\begin{aligned} F(x, y, y') &= x^3 y'^3 - 3x^2 y y'^2 + (x^6 + 3xy^2)y' - y^3 - 2x^5 y \\ &= (xy' - y)^3 + x^6 y' - 2x^5 y = 0\,. \end{aligned}$$

The associated curve defined by $F(x, y, z) = 0$ has the rational parametrization

$$\mathcal{P}(t) = \left(-\frac{t^3x^5 - t^2x^6 + (t-x)^3}{t^3x^5}, -\frac{2t^3x^5 - 2t^2x^6 + (t-x)^3}{t^3x^6} \right).$$

Therefore, the associated differential equation with respect to $\mathcal{P}$ is

$$\omega' = \frac{1}{x^2} \cdot \omega \cdot (2\omega - x)\,,$$

which is a Riccati equation. By applying the algorithm of Kovacic, we can determine a rational general solution of this Riccati equation, namely $\omega(x) = \frac{x}{1+cx^2}$. Hence, the differential equation $F(x, y, y') = 0$ has the rational general solution $y(x) = cx(x + c^2)$.

Observe, that this is just an arbitrary example from the collection of Kamke (1983). In total around 64 percent of the listed ODEs there are AODEs and almost all of them are parametrizable and hence suitable for Algorithm 2 SRGS. As we have seen in Section 3, the remaining ODEs without strong parametrization are either reducible or the corresponding curve has higher genus. For further details see Grasegger et al. (2015).

7. Conclusion

We have presented an algorithm for deciding whether a strong rational general solution of a first-order AODE exists. In the affirmative case the algorithm also computes such a solution. The algorithm in this paper is based on curve parametrizations over the field of rational functions. For parametrizable first-order AODEs even the existence of rational general solutions can be decided. However a full algorithm for determining a rational general solution of the whole class of first-order AODEs is still under investigation.

References

Aroca, J.M., Cano, J., Feng, R., Gao, X.S., 2005. Algebraic general solutions of algebraic ordinary differential equations. In: Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation. ISSAC '05. ACM, New York, NY, USA, pp. 29–36.

Beck, T., Schicho, J., 2008. Curve parametrization over optimal field extensions exploiting the newton polygon. In: Jüttler, B., Piene, R. (Eds.), Geometric Modeling and Algebraic Geometry. Springer, Berlin, pp. 119–140.

Behloul, D., Cheng, S.S., 2011. Computation of rational solutions for a first-order nonlinear differential equation. Electron. J. Differ. Equ. 2011, 1–16 (electronic only).

Carnicer, M.M., 1994. The Poincaré problem in the nondicritical case. Ann. Math. (2) 140 (2), 289–294.

Chen, G., Ma, Y., 2005. Algorithmic reduction and rational general solutions of first order algebraic differential equations. In: Wang, D., Zheng, Z. (Eds.), Differential Equations with Symbolic Computation. Birkhäuser, Basel, pp. 201–212.

Eremenko, A., 1982. Meromorphic solutions of algebraic differential equations. Russ. Math. Surv. 37 (4), 61–95.

Eremenko, A., 1998. Rational solutions of first-order differential equations. Ann. Acad. Sci. Fenn., Math. 23 (1), 181–190.

Feng, R., Gao, X.-S., 2004. Rational general solutions of algebraic ordinary differential equations. In: Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation. ISSAC '04. ACM, New York, NY, USA, pp. 155–162.
Feng, R., Gao, X.-S., 2006. A polynomial time algorithm for finding rational general solutions of first order autonomous odes. J. Symb. Comput. 41 (7), 739–762.
Fuchs, L., 1884. Über Differentialgleichungen, deren Integrale feste Verzweigungspunkte besitzen. Sitzungsber. Acad. 11 (3), 251–273.
Grasegger, G., Vo, N.T., Winkler, F., 2015. Statistical Investigation of First-Order Algebraic ODEs and Their Rational General Solutions. Tech. Rep. 15-19. RISC, Johannes Kepler University Linz.
Hartshorne, R., 1977. Algebraic Geometry. Grad. Texts Math., vol. 52. Springer-Verlag, New York, Berlin, Heidelberg.
Hillgarter, E., Winkler, F., 1997. Points on algebraic curves and the parametrization problem. In: Automated Deduction in Geometry. In: Lect. Notes Comput. Sci. Springer, Berlin, Heidelberg, New York, NY, USA, pp. 189–207.
Hubert, E., 1996. The general solution of an ordinary differential equation. In: Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation. ISSAC '96. ACM, New York, NY, USA, pp. 189–195.
Ince, E., 1956. Ordinary Differential Equations. Dover Books on Mathematics. Dover Publications.
Kamke, E., 1983. Differentialgleichungen: Lösungsmethoden und Lösungen I. B. G. Teubner, Stuttgart.
Kovacic, J.J., 1986. An algorithm for solving second order linear homogeneous differential equations. J. Symb. Comput. 2 (1), 3–43.
Malmquist, J., 1913. Sur les fonctions a un nombre fini de branches définies par les équations différentielles du premier ordre. Acta Math. 36, 297–343.
Matsuda, M., 1980. First Order Algebraic Differential Equations–A Differential Algebraic Approach. Lect. Notes Math., vol. 804. Springer-Verlag, Berlin.
Ngô, L.X.C., Winkler, F., 2010. Rational general solutions of first order non-autonomous parametrizable odes. J. Symb. Comput. 45 (12), 1426–1441.
Ngô, L.X.C., Winkler, F., 2011a. Rational general solutions of parametrizable AODEs. Publ. Math. 79 (3–4), 573–587.
Ngô, L.X.C., Winkler, F., 2011b. Rational general solutions of planar rational systems of autonomous odes. J. Symb. Comput. 46 (10), 1173–1186.
Poincaré, H., 1885. Sur un théorème de M. Fuchs. Acta Math. 7, 1–32.
Ritt, J.F., 1950. Differential Algebra. Colloq. Publ., vol. 33. American Mathematical Society, New York.
Schwarz, F., 2008. Algorithmic Lie Theory for Solving Ordinary Differential Equations. Chapman & Hall/CRC, Boca Raton, FL.
Sendra, J.R., Sevilla, D., 2011. Radical parametrizations of algebraic curves by adjoint curves. J. Symb. Comput. 46 (9), 1030–1038.
Sendra, J.R., Winkler, F., Pérez-Díaz, S., 2008. Rational Algebraic Curves. A Computer Algebra Approach. Springer, Berlin.
Vo, N.T., 2016. Rational and Algebraic Solutions of First-Order Algebraic Ordinary Differential Equations. Ph.D. thesis. Johannes Kepler University Linz.