VL Formal Modeling (Summer semester 2024)

Symbolic Summation and the modeling of sequences

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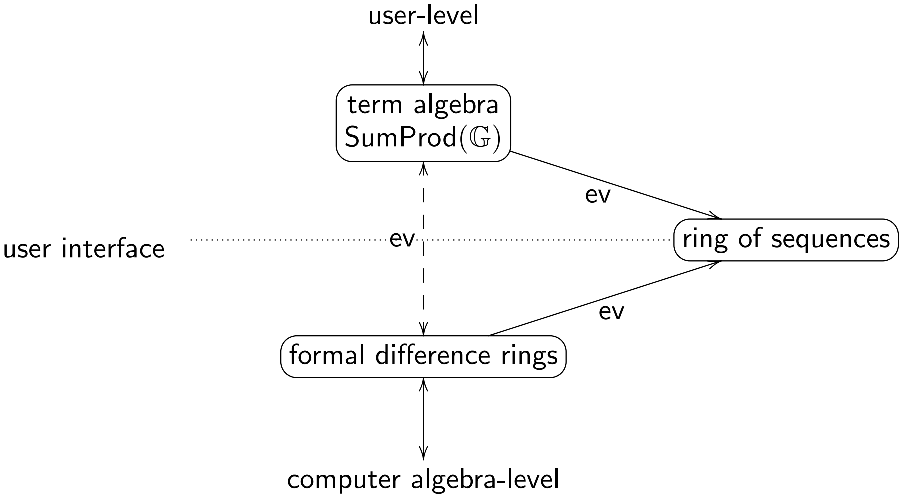
General picture:



General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

We start with the following telescoping problem:

Given an expression $f(k)$ that evaluates to a sequence.
Find an expression $g(k)$ such that the telescoping equation holds:

$$f(k) = g(k+1) - g(k) \tag{1}$$

Suppose we find such an expression $g(k)$. Then we proceed as follows. Summing (1) over k from a to b (and assuming that no poles arise during the evaluation) yields

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a). \tag{2}$$

Note: we could always choose

$$g(k) = \sum_{i=a}^{k-1} f(i) \tag{3}$$

which would turn (2) to the trivial identity $\sum_{k=a}^{b} f(k) = \sum_{k=a}^{b} f(k)$.
Thus we should refine our problem from above:

Find an expression $g(k)$ with (1) where $g(k)$ is simpler than the trivial solution (3).

Indefinite summation of polynomials

We start with one of the most simplest cases: the summand is a polynomial, i.e., $f(x) \in \mathbb{K}[x]$.

The following questions arise:

1. What is the domain of expressions in which we search $g(k)$?
2. How can we calculate a solution $g(k)$ in this solution domain?

As it turns out, the first question can be answered nicely: a solution $g(x)$ exists always in $\mathbb{K}[x]$. For the second question, we will consider two different tactics that are often used in summation packages.

Tactic 1: the classical approach

Note that for indefinite integration of polynomials one can utilize the following well known property: for any $m \in \mathbb{N}$ we have

$$D_x x^m = m\, x^{m-1}$$

which implies

$$\int_a^b x^m dx = \frac{x^{m+1}}{m+1}\Big|_a^b = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

Thus by linearity we can integrate any polynomial by

$$\int_a^b \sum_{m=0}^{d} c_m\, x^m dx = \sum_{m=0}^{d} c_m \int_a^b x^m dx = \sum_{m=0}^{d} \frac{c_m(b^{m+1} - a^{m+1})}{m+1}.$$

For indefinite summation of polynomials we can follow precisely the same classical strategy.

Definition. For any sequence (expression) $g(k)$ we define

$$\Delta g(k) := g(k+1) - g(k).$$

Lemma

For $m \in \mathbb{N}$ we have

$$\boxed{\Delta x^{\underline{m}} = m\, x^{\underline{m-1}}}.$$

Proof.

We have

$$\begin{aligned}\Delta x^{\underline{m}} &= (x+1)^{\underline{m}} - x^{\underline{m}}\\ &= (x+1)x(x-1)\dots(x-m+2) - x(x-1)\dots(x-m+1)\\ &= ((x+1)-(x-m+1))x(x-1)\dots(x-m+2)\\ &= m\, x^{\underline{m-1}}.\end{aligned}$$

□

As a consequence we get

$$\Delta \frac{x^{\underline{m+1}}}{m+1} = x^{\underline{m}}, \quad m \in \mathbb{N}$$

and summing this equation over k from a to b yields

$$\sum_{x=a}^{b} x^{\underline{m}} = \frac{(b+1)^{\underline{m+1}} - a^{\underline{m+1}}}{m+1}.$$

Note that this is nothing else than the continuous version for integration. In particular, for given

$$f(x) = \sum_{m=0}^{d} c_m\, x^{\underline{m}} \in \mathbb{K}[x]$$

with $d \in \mathbb{N}$ it follows that

$$g(x) = \sum_{m=0}^{d} \frac{c_m x^{\underline{m+1}}}{m+1}$$

is a telescoping solution. Furthermore,

$$\sum_{x=a}^{b} f(x) = \sum_{m=0}^{d} c_m \sum_{k=a}^{b} k^{\underline{m}} = \sum_{m=0}^{d} \frac{c_m((b+1)^{\underline{m+1}} - a^{\underline{m+1}})}{m+1}.$$

The only problem is that in many cases one does not have a polynomial given in the representation of falling factorials but in the standard form

$$\sum_{m=0}^{d} \bar{c}_m\, x^m \in \mathbb{K}[x].$$

Luckily one can rewrite a polynomial written in the basis

$$1, x, x^2, \dots, x^d$$

to the representation written in the basis

$$x^{\underline{0}} = 1, x^{\underline{1}} = x, x^{\underline{2}} = x(x-1), \dots, x^{\underline{d}} = x(x-1)\dots(x-d+1)$$

by using the formula

$$x^m = \sum_{k=0}^{m} S(m,k) x^{\underline{k}}$$

where $S(n,k)$ denotes the Stirling numbers of second kind. They can be computed by

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n;$$

Example. Consider the polynomial

$$f(x) = x^4.$$

Using the formulas from above, we get

$$f(x) = x^4 = \sum_{k=0}^{4} S(4,k)x^{\underline{k}} = 0x^{\underline{0}} + 1x^{\underline{1}} + 7x^{\underline{2}} + 6x^{\underline{3}} + 1x^{\underline{4}}.$$

Thus we get

$$\begin{aligned} g(x) &= \frac{1}{2}x^{\underline{2}} + \frac{7}{3}x^{\underline{3}} + \frac{3}{2}x^{\underline{4}} + \frac{1}{5}x^{\underline{5}} \\ &= \frac{1}{30}(x-1)x(2x-1)(3x^2-3x-1). \end{aligned}$$

such that

$$g(x+1) - g(x) = f(x)$$

holds. In particular we get

$$\sum_{k=1}^{n} k^4 = \sum_{k=1}^{n} f(k) = g(n+1) - g(1) = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1).$$

Tactic 2: linear algebra.

We use the following property: for $f(x) \in \mathbb{K}[x]$ there is a $g(x) \in \mathbb{K}[x]$ with (1) where

$$\deg(g) \leq \deg(f) + 1.$$

Thus setting $d := \deg(f) + 1$ for given $f \in \mathbb{K}[x]$ the desired solution has the form

$$g(x) = \sum_{m=0}^{d} g_m \, x^m$$

and we can determine the unknowns $g_0, \dots, g_d \in \mathbb{K}$ by linear algebra as follows.

Example. Take $f(x) = x^4 \in \mathbb{Q}[x]$. With $d = \deg(f) + 1 = 5$ the ansatz

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + g_4 x^4 + g_5 x^5$$

for the unknowns $g_0, g_1, g_2, g_3, g_4, g_5 \in \mathbb{Q}$ is in place. This gives

$$\begin{aligned} x^4 =& \Delta g(x) = g(x+1) - g(x) = 0x^5 \\ &+ 5g_5 x^4 \\ &+ (4g_4 + 10g_5)x^3 \\ &+ (3g_3 + 6g_4 + 10g_5)x^2 \\ &+ (2g_2 + 3g_3 + 4g_4 + 5g_5)x \\ &+ (g_1 + g_2 + g_3 + g_4 + g_5)x^0. \end{aligned}$$

By coefficient comparison this yields the linear system

$$\begin{aligned} &[x^4] & 1 =& 5g_5 \\ &[x^3] & 0 =& 4g_4 + 10g_5 \\ &[x^2] & 0 =& 3g_3 + 6g_4 + 10g_5 \\ &[x^1] & 0 =& 2g_2 + 3g_3 + 4g_4 + 5g_5 \\ &[x^0] & 0 =& g_1 + g_2 + g_3 + g_4 + g_5 \end{aligned}$$

which is already in triangular form.

Thus we can read off the solution

$$g_5 = \frac{1}{5}, \quad g_4 = -\frac{1}{2}, \quad g_3 = \frac{1}{3}, \quad g_2 = 0, \quad g_1 = -\frac{1}{30}, \quad g_0 = c$$

with $c \in \mathbb{Q}$. In particular, we can choose $c = 0$ and obtain

$$g(x) = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30} = \frac{1}{30}(x-1)x(2x-1)\left(3x^2 - 3x - 1\right).$$

To this end, we continue as in the previous example and get the desired result.

More general summation objects for indefinite and definite summation

Clearly, the first tactic is very elegant, but it works only for the special case of polynomial summation. For the second tactic one has to work more (i.e., has to solve in addition a linear system), but it turns out to be more general. More precisely, one can carry over these ideas to a rather general setting that works not only for the polynomial ring $\mathbb{Q}[x]$ but in more general rings called $R\Pi\Sigma$-difference rings that have been implemented within the summation package `Sigma`. In the following all technical details are omitted and we proceed with a concrete example.

Example. We want to sum

$$\sum_{k=0}^{n} H_k.$$

In order to accomplish this task, we take

$$f(k) = H_k$$

and search for

$$g(k) \in \mathbb{Q}(k)[H_k]$$

with

$$f(k) = g(k+1) - g(k).$$

Here we can use a similar tactic as used in the case of polynomial summation. Namely, summation theory tells us that any such solution $g(k)$ has the property

$$\deg(g) \leq \deg(f) + 1 = 1 + 1 = 2.$$

As a consequence we can make the ansatz

$$g(k) = g_0(k)H_k^0 + g_1(k)H_k^1 + g_2(k)H_k^2$$

with $g_0(k), g_1(k), g_2(k) \in \mathbb{Q}(k)$.

Using recursive algorithms and linear system solving (details are skipped here) we find

$$\begin{aligned} g_0(k) &= -k \\ g_1(k) &= k \\ g_2(k) &= 0, \end{aligned}$$

i.e.,

$$g(k) = -k + kH_k + 0H_k^2.$$

Hence summing the telescoping equation over k from 0 to n gives

$$\sum_{k=0}^{n} H_k = g(n+1) - g(0) = (n+1)H_{n+1} - (n+1) = -n + (1+n)H_n.$$

The above machinery can be carried out within the summation package `Sigma`. After loading it into Mathematica

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-JKU

one can insert the above sum

In[2]:= $\mathbf{mySum = SigmaSum[SigmaHNumber[k], \{k, 0, 1\}]}$

Out[2]= $\sum_{k=0}^{n} H_k$

and can apply the command

In[3]:= $\mathbf{SigmaReduce[mySum]}$

Out[3]= $-n + (1 + n)H_n$

In general one can insert, e.g., a sum of the form

$$\sum_{k=l}^{n} f(k)$$

with $l \in \mathbb{N}$ where $f(k)$ itself is given in terms of indefinite nested sums defined over hypergeometric products.

Definition

Let $\mathbb{K}$ be a field. A product $\prod_{j=l}^{k} f(j)$, $l \in \mathbb{N}$, is called **hypergeometric in** k **over** $\mathbb{K}$ if $f(x) \in \mathbb{K}(x)$ is a rational function where the numerator and denominator of $f(j)$ are nonzero for all $j \in \mathbb{Z}$ with $j \geq l$. An **expression in terms indefinite of nested sums over hypergeometric products in** k **over** $\mathbb{K}$ is composed recursively by the three operations $(+, -, \cdot)$ with

- elements from the rational function field $\mathbb{K}(k)$,
- hypergeometric products in k over $\mathbb{K}$,
- and sums of the form $\sum_{j=l}^{k} f(j)$ with $l \in \mathbb{N}$ where $f(j)$ is an expression in terms of indefinite nested sums over hypergeometric products in j over $\mathbb{K}$; here it is assumed that the evaluation of $f(j)$ for all $j \geq l$ does not introduce any poles.

In[4]:= $\mathbf{mySum} =$

$\mathbf{SigmaSum[SigmaPower[-1, k]SigmaBinomial[n, k]SigmaHNumber[k], \{k, a, b\}]}$

Out[4]= $$\sum_{k=a}^{b}(-1)^k\binom{n}{k}H_k$$

In[5]:= $\mathbf{SigmaReduce[mySum]}$

Out[5]= $$\left(\frac{(a-n)(-1+a-n)}{an^2}+\frac{(-1+a-n)H_a}{n}\right)(-1)^{1+a}\binom{n}{-1+a}+\left(\frac{-b+n}{n^2}+\frac{(-b+n)H_b}{n}\right)(-1)^b\binom{n}{b}$$

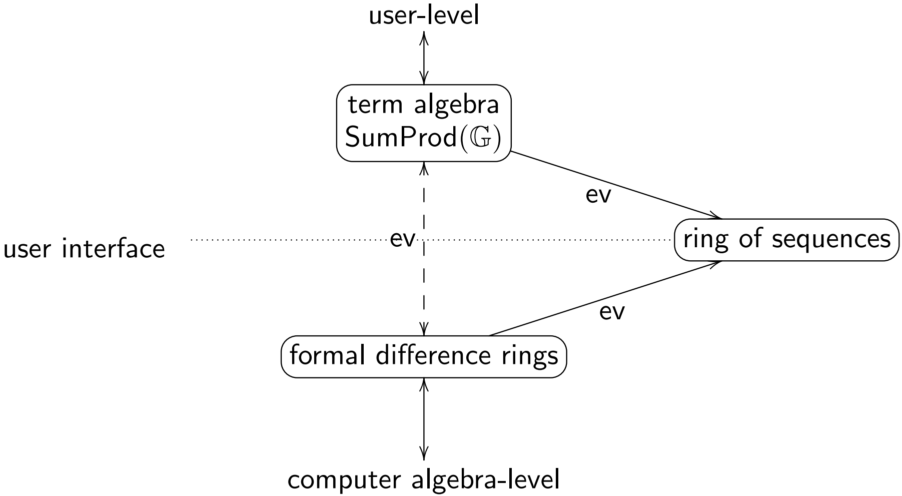
In[6]:= $\mathbf{mySum = SigmaSum[SigmaSum[SigmaBinomial[n, k], \{k, 0, r\}]^2, \{r, 0, b\}]}$

Out[6]= $$\sum_{r=0}^{b}\left(\sum_{k=0}^{r}\binom{n}{k}\right)^2$$

In[7]:= $\mathbf{SigmaReduce[mySum]}$

Out[7]= $$(-b+n)\binom{n}{b}\sum_{i_1=0}^{b}\binom{n}{i_1}+\frac{1}{2}(2+2b-n)\left(\sum_{i_1=0}^{b}\binom{n}{i_1}\right)^2-\frac{1}{2}n\sum_{i_1=0}^{b}\binom{n}{i_1}^2$$

General picture:



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The ground field (throughout this talk): $\mathbb{G} = \mathbb{K}(x)$

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- For any element $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x]$ where $q \neq 0$ and p, q being coprime we define

$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k) = 0 \\ \frac{p(k)}{q(k)} & \text{if } q(k) \neq 0. \end{cases}$$

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- We define $L(f)$ to be the minimal value $\delta \in \mathbb{N}$ such that $q(k) \neq 0$ holds for all $k \geq \delta$; further,

$$Z(f) = \max(L(\tfrac{1}{p}), L(\tfrac{1}{q})) \qquad \text{if } f \neq 0.$$

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Example: For

$$f = \frac{p}{q} = \frac{x-4}{(x-3)(x-1)}$$

we get

$$(\mathrm{ev}(f,n))_{n\geq 0} = (-\tfrac{4}{3}, \underline{0}, 2, \underline{0}, 0, \tfrac{1}{8}, \dots) \in \mathbb{Q}^{\mathbb{N}}$$

For $n \geq L(f) = 4$ no poles arise;

for $n \geq Z(f) = \max(L(\frac{1}{p}), L(\frac{1}{q})) = \max(4,5) = 5$ no zeroes arise.

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- We define

$$\mathcal{R} = \{r \in \mathbb{K} \setminus \{1\} \mid r \text{ is a root of unity}\}$$

with the function $\operatorname{ord} : \mathcal{R} \to \mathbb{Z}_{\geq 1}$ where

$$\operatorname{ord}(r) = \min\{n \in \mathbb{Z}_{\geq 1} \mid r^n = 1\}.$$

$\mathbb{G} \quad \longrightarrow \quad$ SumProd($\mathbb{G}$) (nested sums over hypergeometric products)

Let $\circledwedge$, $\oplus$, $\odot$, Sum, Prod and RPow be operations with the signatures

$$\begin{array}{lll}
\circledwedge: & \text{SumProd}(\mathbb{G}) \times \mathbb{Z} & \to \text{SumProd}(\mathbb{G}) \\
\oplus: & \text{SumProd}(\mathbb{G}) \times \text{SumProd}(\mathbb{G}) & \to \text{SumProd}(\mathbb{G}) \\
\odot: & \text{SumProd}(\mathbb{G}) \times \text{SumProd}(\mathbb{G}) & \to \text{SumProd}(\mathbb{G}) \\
\text{Sum}: & \mathbb{N} \times \text{SumProd}(\mathbb{G}) & \to \text{SumProd}(\mathbb{G}) \\
\text{Prod}: & \mathbb{N} \times \text{SumProd}(\mathbb{G}) & \to \text{SumProd}(\mathbb{G}) \\
\text{RPow}: & \mathcal{R} & \to \text{SumProd}(\mathbb{G}).
\end{array}$$

Prod$^*(\mathbb{G})$= the smallest set that contains 1 with the following properties:

1. If $r \in \mathcal{R}$ then RPow$(r) \in$ Prod$^*(\mathbb{G})$.
2. If $f \in \mathbb{G}^*$ and $l \in \mathbb{N}$ with $l \geq Z(f)$ then Prod$(l, f) \in$ Prod$^*(\mathbb{G})$.
3. If $p, q \in$ Prod$^*(\mathbb{G})$ then $p \odot q \in$ Prod$^*(\mathbb{G})$.
4. If $p \in$ Prod$^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p^{\circledwedge} z \in$ Prod$^*(\mathbb{G})$.

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Furthermore, we define

$$\Pi(\mathbb{G}) = \{\mathrm{RPow}(r) \mid r \in \mathcal{R}\} \cup \{\mathrm{Prod}(l, f) \mid f \in \mathbb{G}, l \in \mathbb{N}\}.$$

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Example: In $\mathbb{G} = \mathbb{Q}(x)$ we get

$$P = (\underbrace{\mathsf{Prod}(1, x)}_{\in \Pi(\mathbb{G})}{}^{\circledwedge}(-2)) \odot \underbrace{\mathsf{RPow}(-1)}_{\Pi(\mathbb{G})} \in \mathsf{Prod}^*(\mathbb{G}).$$

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SumProd($\mathbb{G}$) = the smallest set containing $\mathbb{G} \cup \text{Prod}^*(\mathbb{G})$ with:

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Furthermore, the **set of nested sums over hypergeometric products** is given by

$$\Sigma(\mathbb{G}) = \{\text{Sum}(l, f) \mid l \in \mathbb{N} \text{ and } f \in \text{SumProd}(\mathbb{G})\}$$

and the **set of nested sums and hypergeometric products** is given by

$$\Sigma\Pi(\mathbb{G}) = \Sigma(\mathbb{G}) \cup \Pi(\mathbb{G}).$$

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Example

With $\mathbb{G} = \mathbb{K}(x)$ we get, e.g., the following expressions:

$$
\begin{aligned}
E_1 &= \text{Sum}(1, \text{Prod}(1, x)) \in \Sigma(\mathbb{G}) \subset \text{SumProd}(\mathbb{G}),\\
E_2 &= \text{Sum}(1, \tfrac{1}{x+1} \odot \text{Sum}(1, \tfrac{1}{x^3}) \odot \text{Sum}(1, \tfrac{1}{x})) \in \Sigma(\mathbb{G}) \subset \text{SumProd}(\mathbb{G}),\\
E_3 &= (E_1 \oplus E_2) \odot E_1 \in \text{SumProd}(\mathbb{G}).
\end{aligned}
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$\mathrm{ev} : \mathbb{G} \times \mathbb{N} \to \mathbb{K} \qquad \longrightarrow \qquad \mathrm{ev} : \mathsf{SumProd}(\mathbb{G}) \times \mathbb{N} \to \mathbb{K}$

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$$\begin{aligned}
\text{ev}(f^{\textcircled{\wedge}}k, n) &:= \text{ev}(f, n)^k, \\
\text{ev}(f \oplus g, n) &:= \text{ev}(f, n) + \text{ev}(g, n), \\
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$$\begin{aligned}
\mathrm{ev}(\mathsf{RPow}(r), n) &:= \prod_{i=1}^{n} r = r^n, \\
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Note: $\Pi(\mathbb{G})$ defines all hypergeometric products (which evaluate to sequences with non-zero entries).

ev applied to $f \in \mathsf{SumProd}(\mathbb{G})$ represents a sequence.
f can be considered as a simple program and $\mathrm{ev}(f, n)$ with $n \in \mathbb{N}$ executes it (like an interpreter/compiler) yielding the nth entry of the represented sequence.

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For $E_i \in \mathsf{SumProd}(\mathbb{K}(x))$ with $i = 1, 2, 3$ we get

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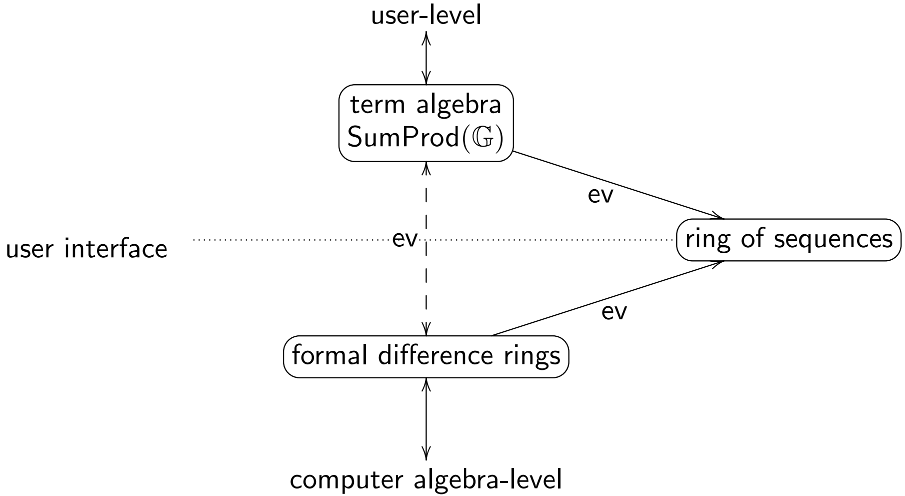
$$E_3(n) = (E_1(n) + E_2(n)) E_1(n)$$

General picture:



Definition

An expression $A \in \mathsf{SumProd}(\mathbb{G})$ is in **reduced representation** if

$$A = (f_1 \odot P_1) \oplus (f_2 \odot P_2) \oplus \cdots \oplus (f_r \odot P_r) \tag{4}$$

with $f_i \in \mathbb{G}^*$ and

$$P_i = (a_{i,1} \circledwedge z_{i,1}) \odot (a_{i,2} \circledwedge z_{i,2}) \odot \cdots \odot (a_{i,n_i} \circledwedge z_{i,n_i})$$

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for $1 \leq i \leq r$ where

- $a_{i,j} = \mathsf{Sum}(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in \mathsf{SumProd}(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z}_{\geq 1}$,
- $a_{i,j} = \mathsf{Prod}(l_{i,j}, f_{i,j})$ with $l_{i,j} \in \mathbb{N}$, $f_{i,j} \in \mathsf{Prod}^*(\mathbb{G})$ and $z_{i,j} \in \mathbb{Z} \setminus \{0\}$,
- $a_{i,j} = \mathsf{RPow}(f_{i,j})$ with $f_{i,j} \in \mathcal{R}$ and $1 \leq z_{i,j} < \mathrm{ord}(r_{i,j})$

such that the following properties hold:

1. for each $1 \leq i \leq r$ and $1 \leq j < j' < n_i$ we have $a_{i,j} \neq a_{i,j'}$;
2. for each $1 \leq i < i' \leq r$ with $n_i = n_j$ there does not exist a $\sigma \in S_{n_i}$ with $P_{i'} = (a_{i,\sigma(1)} {}^{\textcircled{\wedge}} z_{i,\sigma(1)}) \odot (a_{i,\sigma(2)} {}^{\textcircled{\wedge}} z_{i,\sigma(2)}) \odot \cdots \odot (a_{i,\sigma(n_i)} {}^{\textcircled{\wedge}} z_{i,\sigma(n_i)})$.

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Lemma

For any $A \in \mathsf{SumProd}(\mathbb{G})$, there is a $B \in \mathsf{SumProd}(\mathbb{G})$ in sum-product reduced representation and $\lambda \in \mathbb{N}$ such that

$$A(n) = B(n) \quad \forall n \geq \lambda.$$

Key-Definitions: Let $W \subseteq \Sigma\Pi(\mathbb{G})$.

SumProd$(W, \mathbb{G}) =$ the set of elements from SumProd$(\mathbb{G})$ which are in reduced representation and the arising sums/products are taken from W.

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- W is called **shift-closed over** $\mathbb{G}$ if for any $A \in$ SumProd$(W, \mathbb{G})$, $s \in \mathbb{Z}$ there are $B \in$ SumProd$(W, \mathbb{G})$ and $\delta \in \mathbb{N}$ such that

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- W is called **shift-stable over** $\mathbb{G}$ if for any product or sum in W the multiplicand or summand is built by sums and products from W.
- W is called **canonical reduced over** $\mathbb{G}$ if for any $A, B \in$ SumProd$(W, \mathbb{G})$ with
$$A(n) = B(n) \quad \forall n \geq \delta$$
for some $\delta \in \mathbb{N}$ the following holds: A and B are the same up to permutations of the operands in $\oplus$ and $\odot$.

Definition

$W \subseteq \Sigma\Pi(\mathbb{G})$ is called σ-**reduced over** $\mathbb{G}$ if

1. the elements in W are in sum-product reduced form,
2. W is shift-stable (and thus shift-closed) and
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In particular, $A \in \mathsf{SumProd}(W, \mathbb{G})$ is called σ-**reduced (w.r.t.** W**)** if W is σ-reduced over $\mathbb{G}$.

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Problem SigmaReduce: Compute a σ-reduced representation

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.
Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$,
$B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$
such that for all $1 \leq i \leq r$ we get

$$A_i(n) = B_i(n) \quad n \geq \delta_i.$$

- **Canonical representation in term algebras**

$$\begin{tikzcd} A_1 \arrow[d] \\ B_1 \end{tikzcd}$$

in $\mathsf{SumProd}(\mathbb{G})$

σ-reduced

$\forall n \geq \delta \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n)$

- **Canonical representation in term algebras**

$$\begin{array}{ccccc} A_1 & & & A_2 & \\ \downarrow & & & \downarrow & \text{in } \mathsf{SumProd}(\mathbb{G}) \\ B_1 & & \sigma\text{-reduced} & B_2 & \end{array}$$

$\forall n \geq \delta \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \qquad \mathsf{ev}(A_2, n) = \mathsf{ev}(B_2, n)$

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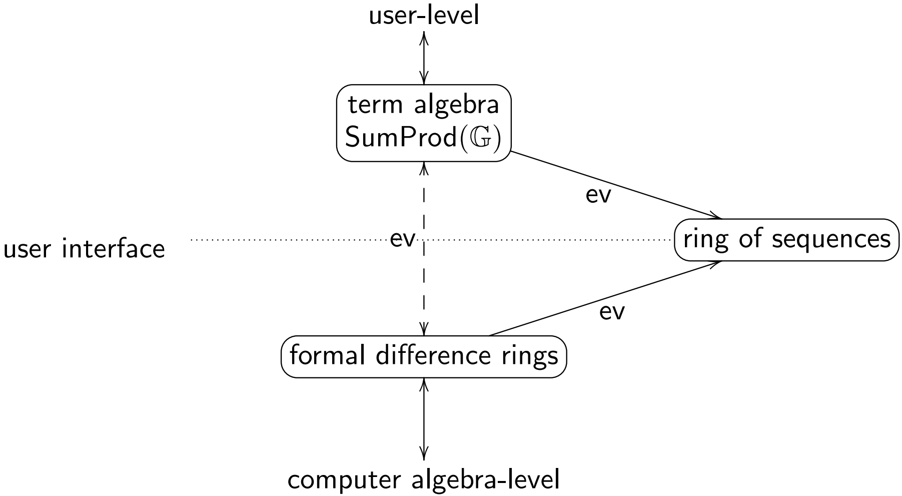
$$

$$\forall n \geq \delta \;\; \mathsf{ev}(A_1, n) = \mathsf{ev}(B_1, n) \qquad = \qquad \mathsf{ev}(A_2, n) = \mathsf{ev}(B_2, n)$$

$$\Updownarrow \qquad \text{canonical simplifier}$$

$$B_1 = B_2$$

General picture:



General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

Represent $H = \mathsf{Sum}(1, \frac{1}{x}) \in \mathsf{SumProd}(\mathbb{G})$ with

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2. an evaluation function

$$\begin{aligned} \mathrm{ev}': \quad & \mathbb{Q}(x) \times \mathbb{N} && \to \quad \mathbb{Q} \\ & \left(\tfrac{p(x)}{q(x)}, n\right) && \mapsto \quad \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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$$\mathrm{ev} : \quad \mathbb{Q}(x)[s] \times \mathbb{N} \quad \to \quad \mathbb{Q}$$

$$\mathbf{ev(s, n) = H_n}$$

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2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

$$\begin{array}{rlcl} \mathrm{ev}' : & \mathbb{Q}(x) \times \mathbb{N} & \to & \mathbb{Q} \\ & (\frac{p(x)}{q(x)}, n) & \mapsto & \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

$$\begin{array}{rlcl} \mathrm{ev} : & \mathbb{Q}(x)[s] \times \mathbb{N} & \to & \mathbb{Q} \\ & \left(\sum_{i=0}^{d} f_i\, s^i, n\right) & \mapsto & \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) H_n^i \end{array} \qquad \mathbf{ev(s,n) = H_n}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Represent $H = \mathsf{Sum}(1, \frac{1}{x}) \in \mathsf{SumProd}(\mathbb{G})$ with

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Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \qquad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$

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$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

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It is an injective ring homomorphism (ring embedding):

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$$\begin{array}{rlcl} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \displaystyle\sum_{i=0}^{d} f_i\, s^i & \mapsto & \displaystyle\sum_{i=0}^{d} \sigma'(f_i)\Big(s + \tfrac{1}{x+1}\Big)^i \end{array} \qquad \begin{array}{l} s \mapsto s + \frac{1}{x+1} \\ H_{n+1} = H_n + \frac{1}{n+1} \end{array}$$

Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Represent $H = \mathsf{Sum}(1, \frac{1}{x}) \in \mathsf{SumProd}(\mathbb{G})$ with

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ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = H_n + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

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$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Represent $H = \mathsf{Sum}(1, \frac{1}{x}) \in \mathsf{SumProd}(\mathbb{G})$ with

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τ is an injective difference ring homomorphism:

$$\begin{tikzcd}
\mathbb{K}(x)[s] \arrow[r, "\sigma"] \arrow[d, "\tau"] \arrow[dr, phantom, "="] & \mathbb{K}(x)[s] \arrow[d, "\tau"] \\
\mathbb{K}^{\mathbb{N}}/\sim \arrow[r, "S"] & \mathbb{K}^{\mathbb{N}}/\sim
\end{tikzcd}$$

Represent $H = \mathsf{Sum}(1, \frac{1}{x}) \in \mathsf{SumProd}(\mathbb{G})$ with

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τ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle H_n \rangle_{n \geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

Summary: we rephrase $H \in \mathsf{SumProd}(\mathbb{G})$ as element h in a formal difference ring. More precisely, we will design

- a ring $\mathbb{A}$ with $\mathbb{A} \supseteq \mathbb{G} \supseteq \mathbb{K}$ in which H can be represented by $h \in \mathbb{A}$;
- an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ such that $H(n) = \mathrm{ev}(h, n)$ holds for sufficiently large $n \in \mathbb{N}$;
- a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ which models $H(n+1)$ with $\sigma(h)$.

A hypergeometric APS-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

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Sk!=(k+1)k!

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Sk!=(k+1)k! $\quad\leftrightarrow\quad \sigma(p_1) = (x+1)p_1$

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}]}$$

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 & & \vdots & \\
 & & \sigma(p_e) = a_e p_e & a_e \in \mathbb{K}(x)^* \\
\mathbf{(-1)^k} & \leftrightarrow & \boldsymbol{\sigma(z) = -z} & \mathbf{z^2 = 1}
\end{array}$$

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γ is a primitive λth root of unity

A hypergeometric APS-extension of $(\mathbb{K}(x),\sigma)$ is

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$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1]}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\begin{array}{rcll}
 & & \sigma(x)=x+1 & \\
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 & & \vdots & \\
 & & \sigma(p_e)=a_ep_e & a_e\in\mathbb{K}(x)^* \\
\gamma\text{ is a primitive }\lambda\text{th root of unity}\quad \gamma^{\mathbf{k}} & \leftrightarrow & \sigma(\mathbf{z})=\gamma\,\mathbf{z} & \mathbf{z}^{\lambda}=\mathbf{1} \\
H_{k+1}=H_k+\frac{1}{k+1} & \leftrightarrow & \sigma(s_1)=s_1+\frac{1}{x+1} &
\end{array}$$

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$$\sigma(x)=x+1$$

hypergeometric products $\leftrightarrow$

$$\begin{aligned}\sigma(p_1)&=a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\ \sigma(p_2)&=a_2p_2 && a_2\in\mathbb{K}(x)^*\\ &\vdots\\ \sigma(p_e)&=a_ep_e && a_e\in\mathbb{K}(x)^*\end{aligned}$$

γ is a primitive λth root of unity $\quad\gamma^{\mathbf{k}}\quad\leftrightarrow\quad\boldsymbol{\sigma(\mathbf{z})=\gamma\,\mathbf{z}}\qquad\mathbf{z}^{\lambda}=\mathbf{1}$

(nested) sum $\leftrightarrow$ $\sigma(s_1)=s_1+f_1\qquad f_1\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]$

A hypergeometric APS-extension of $(\mathbb{K}(x),\sigma)$ is

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hypergeometric products	$\leftrightarrow$	$\sigma(p_1)=a_1\,p_1$ $\sigma(p_2)=a_2p_2$ $\vdots$ $\sigma(p_e)=a_ep_e$	$a_1\in\mathbb{K}(x)^*$ $a_2\in\mathbb{K}(x)^*$ $a_e\in\mathbb{K}(x)^*$
γ is a primitive λth root of unity ${\color{blue}\gamma^{\mathbf{k}}}$	$\leftrightarrow$	${\color{blue}\sigma(\mathbf{z})=\gamma\,\mathbf{z}}$	${\color{blue}\mathbf{z}^{\lambda}=\mathbf{1}}$
(nested) sum	$\leftrightarrow$	$\sigma(s_1)=s_1+f_1$ $\sigma(s_2)=s_2+f_2$	$f_1\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]$ $f_2\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1]$

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(nested) sum $\leftrightarrow$

$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

Definition (Evaluation function)

Take $(\mathbb{A}, \sigma)$ with a subfield $\mathbb{K}$ of $\mathbb{A}$ with $\sigma|_{\mathbb{K}} = \mathrm{id}$.

1. $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{K}$ is called **evaluation function** for $(\mathbb{A}, \sigma)$ if for all $f, g \in \mathbb{A}$, $c \in \mathbb{K}$ and $l \in \mathbb{Z}$ there exists a $\lambda \in \mathbb{N}$ with

$$\forall n \geq \lambda : \mathrm{ev}(c, n) = c, \tag{5}$$

$$\forall n \geq \lambda : \mathrm{ev}(f + g, n) = \mathrm{ev}(f, n) + \mathrm{ev}(g, n), \tag{6}$$

$$\forall n \geq \lambda : \mathrm{ev}(f\, g, n) = \mathrm{ev}(f, n)\ \mathrm{ev}(g, n), \tag{7}$$

$$\forall n \geq \lambda : \mathrm{ev}(\sigma^l(f), n) = \mathrm{ev}(f, n + l). \tag{8}$$

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$$\forall n\geq\lambda:\mathrm{ev}(\sigma^{l}(f),n)=\mathrm{ev}(f,n+l). \tag{8}$$

2. $L:\mathbb{A}\to\mathbb{N}$ is called o-**function** if for any $f,g\in\mathbb{A}$ with $\lambda=\max(L(f),L(g))$ the properties (6) and (7) hold and for any $f\in\mathbb{A}$ and $l\in\mathbb{Z}$ with $\lambda=L(f)+\max(0,-l)$ property (8) holds.

Connection between SumProd($\mathbb{G}$) and hypergeometric APS-extension

- **Observation 1:** Given $\{T_1, \dots, T_e\} \subseteq \Sigma\Pi(\mathbb{G})$, one can construct a hypergeometric APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with ev and L such that there are $a_1, \dots, a_e \in \mathbb{E}$ and $\delta_1, \dots, \delta_e$ with $\mathrm{ev}(a_i, n) = T_i(n)$.

Connection between SumProd($\mathbb{G}$) and hypergeometric APS-extension

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In particular, if $f\in\mathbb{E}\setminus\{0\}$, then we can take the "unique"
$0\neq F\in$ SumProd($\{T_1,\dots,T_e\},\mathbb{G}$) with $F(n)=\mathrm{ev}(f,n)$ for all $n\geq L(f)$.

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Definition

For $f \in \mathbb{E}$ we also write expr$(f) = F$ for this particular F.

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Example

For $f = x + \frac{x+1}{x} s^4 \in \mathbb{Q}(x)[s]$ we obtain

$$\text{expr}(f) = F = x \oplus (\tfrac{x+1}{x} \odot (\text{Sum}(1, \tfrac{1}{x})^{\wedge}4) \in \text{Sum}(\mathbb{Q}(x)))$$

with $F(n) = \mathrm{ev}(f, n)$ for all $n \geq 1$.

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Difference ring theory in action

Let $(\mathbb{E}, \sigma)$ be a hypergeometric APS-extension of $(\mathbb{G}, \sigma)$ with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ and let $\tau : \mathbb{E} \to \mathbb{K}^{\mathbb{N}}/\sim$ be the $\mathbb{K}$-homomorphism given by

$$\tau(f) = (\mathrm{ev}(f, n))_{n \geq 0}.$$

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Lemma

Let $W = \{T_1, \dots, T_e\} \in \Sigma\Pi(\mathbb{G})$ *with* $T_i =$ *expr*(t_i). *Then:*

W *is canonical reduced* $\quad\Leftrightarrow\quad$ τ *is injective.*

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Let $W = \{T_1, \ldots, T_e\} \in \Sigma\Pi(\mathbb{G})$ with $T_i = expr(t_i)$. Then:

$$W \textit{ is canonical reduced} \quad \Leftrightarrow \quad \tau \textit{ is injective.}$$

Using difference ring theory we get the following crucial property:

Theorem

$$\tau \textit{ is injective} \quad \Leftrightarrow \quad \mathrm{const}_\sigma \mathbb{E} = \mathbb{K}.$$

Example

For our difference field $\mathbb{G} = \mathbb{K}(x)$ with $\sigma(x) = x + 1$ and $\mathrm{const}_\sigma\mathbb{K} = \mathbb{K}$ we have $\mathrm{const}_\sigma\mathbb{K}(x) = \mathbb{K}$.

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A hypergeometric APS-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ is called **hypergeometric $R\Pi\Sigma$-extension** if

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Theorem

Let $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ be in sum-product reduced representation and shift-stable, i.e., for each $1 \leq i \leq e$ the arising sums and products in T_i are contained in $\{T_1, \dots, T_{i-1}\}$. Then the following is equivalent:

1. *There is a hypergeometric $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \dots \langle t_e \rangle$ equipped with an evaluation function ev with $T_i = \textit{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.*
2. *W is σ-reduced over $\mathbb{G}$.*

This yields a strategy (actually the only strategy for shift-stable sets).

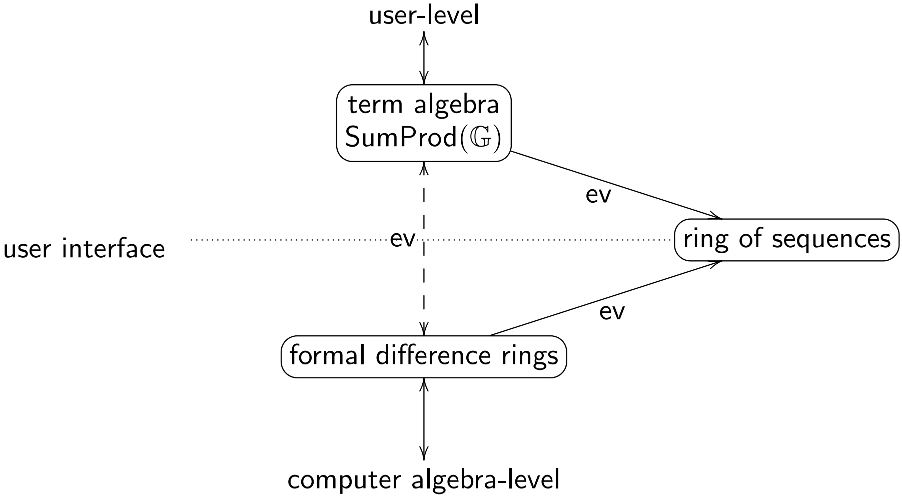
A Strategy to solve Problem SigmaReduce

Given: $A_1, \dots, A_u \in$ SumProd($\mathbb{G}$) with $\mathbb{G} = \mathbb{K}(x)$.

Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in$ SumProd$(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \dots \langle t_e \rangle$ equipped with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \dots, a_u \in \mathbb{E}$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ with
$$A_i(n) = \mathrm{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{12}$$
2. Set $W = \{T_1, \dots, T_e\}$ with $T_i :=$ expr$(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.
3. Set $B_i :=$ expr$(a_i) \in$ SumProd$(W, \mathbb{G})$ for $1 \leq i \leq u$.
4. Return W, $(B_1, \dots, B_u)$ and $(\delta_1, \dots, \delta_u)$.

General picture:



General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

A hypergeometric APS-extension of $(\mathbb{K}(x), \sigma)$ is

- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \dots}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)^* \end{aligned}$$

γ is a primitive λth root of unity $\quad \gamma^{\mathbf{k}} \leftrightarrow$

$$\boldsymbol{\sigma(z) = \gamma\, z} \qquad \mathbf{z^{\lambda} = 1}$$

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$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

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such that $\mathrm{const}_\sigma\mathbb{E} = \mathbb{K}$

Represent sums (extension of Karr's result, 1981)

- Let $(\mathbb{A}, \sigma)$ be a difference ring with constant set

$$\mathrm{const}_\sigma \mathbb{A} := \{k \in \mathbb{A} | \sigma(k) = k\}.$$

Note 1: $\mathrm{const}_\sigma \mathbb{A}$ is a ring that contains $\mathbb{Q}$

Note 2: We always take care that $\mathrm{const}_\sigma \mathbb{A}$ is a field

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2. $\boxed{\exists g \in \mathbb{A}: \sigma(g) = g + f}$: No need for a Σ^*-extension!

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There are 3 cases:

1. $\boxed{\nexists g \in \mathbb{A} \setminus \{0\} \nexists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n\, g}$: $(\mathbb{A}[t, \frac{1}{t}]), \sigma)$ is a Π-ext. of $(\mathbb{A}, \sigma)$

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1. $\boxed{\nexists g \in \mathbb{A} \setminus \{0\} \nexists n \in \mathbb{Z} \setminus \{0\} : \sigma(g) = a^n\, g}$: $(\mathbb{A}[t, \frac{1}{t}]), \sigma)$ is a Π-ext. of $(\mathbb{A}, \sigma)$
2. $\boxed{\exists g \in \mathbb{A} \setminus \{0\} : \sigma(g) = a\, g}$: No need for a Π-extension!

Represent products (extension of Karr's result, 1981)

- Let $(\mathbb{A},\sigma)$ be a difference ring with constant field
$$\text{const}_\sigma\mathbb{A} := \{k \in \mathbb{A} | \sigma(k) = k\}.$$
- Take the ring of Laurent polynomials $\mathbb{A}[t, \frac{1}{t}]$.
- Extend the shift operator s.t.
$$\sigma(t) = a\,t \qquad \text{for some } a \in \mathbb{A}^*.$$

Then $\text{const}_\sigma\mathbb{A}[t, t^{-1}] = \text{const}_\sigma\mathbb{A}$ iff
$$\nexists g \in \mathbb{A} \setminus \{0\}\, \nexists n \in \mathbb{Z} \setminus \{0\} : \quad \boxed{\sigma(g) = a^n\, g}$$

There are 3 cases:

1. $\boxed{\nexists g \in \mathbb{A}\setminus\{0\} \nexists n \in \mathbb{Z}\setminus\{0\} : \sigma(g) = a^n\, g}$: $(\mathbb{A}[t, \frac{1}{t}]), \sigma)$ is a Π-ext. of $(\mathbb{A},\sigma)$
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3. $\boxed{\exists g \in \mathbb{A}\setminus\{0\} : \sigma(g) = a^n\, g \text{ only for } n \in \mathbb{Z} \setminus \{0, 1\}}$: ☹

The hypergeometric case

- Take the difference field $(\mathbb{K}(x), \sigma)$ with $\sigma|_{\mathbb{K}} = \text{id}$ and $\sigma(x) = x + 1$.
- Let $\alpha_1, \dots, \alpha_r \in \mathbb{K}(x)^*$

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such that for $1 \leq i \leq r$ there are $g_i \in \mathbb{E}^*$ with

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with

- $\frac{\sigma(t_i)}{t_i} \in \mathbb{K}(x)^*$ for $1 \leq i \leq e$
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Note: There are similar results for the q-rational, multi-basic and mixed case

A hypergeometric $R\Pi\Sigma$-extension of $(\mathbb{K}(x),\sigma)$ is

- a ring

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x) = x+1$$

hypergeometric products $\leftrightarrow$

$$\begin{aligned}\sigma(p_1) &= a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\ \sigma(p_2) &= a_2p_2 && a_2\in\mathbb{K}(x)^*\\ &\vdots\\ \sigma(p_e) &= a_ep_e && a_e\in\mathbb{K}(x)^*\end{aligned}$$

γ is a primitive λth root of unity $\quad \gamma^{\mathbf{k}} \quad \leftrightarrow$

$$\boldsymbol{\sigma(\mathbf{z}) = \gamma\,\mathbf{z}} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$$

(nested) sum $\leftrightarrow$

$$\begin{aligned}\sigma(s_1) &= s_1+f_1 && f_1\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]\\ \sigma(s_2) &= s_2+f_2 && f_2\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1]\\ \sigma(s_3) &= s_3+f_3 && f_3\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]\\ &\vdots\end{aligned}$$

such that $\mathrm{const}_{\sigma}\mathbb{E} = \mathbb{K}$

This yields a strategy (actually the only strategy for shift-stable sets).

A Strategy to solve Problem SigmaReduce

Given: $A_1, \dots, A_u \in \mathsf{SumProd}(\mathbb{G})$ with $\mathbb{G} = \mathbb{K}(x)$.

Find: a σ-reduced set $W = \{T_1, \dots, T_e\} \subset \Sigma\Pi(\mathbb{G})$ with $B_1 \dots, B_u \in \mathsf{SumProd}(W, \mathbb{G})$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ such that $A_i(n) = B_i(n)$ holds for all $n \geq \delta_i$ and $1 \leq i \leq r$.

1. Construct $R\Pi\Sigma$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}, \sigma)$ with $\mathbb{E} = \mathbb{G}\langle t_1 \rangle \dots \langle t_e \rangle$ equipped with $\mathrm{ev} : \mathbb{E} \times \mathbb{N} \to \mathbb{K}$ such that we get $a_1, \dots, a_u \in \mathbb{E}$ and $\delta_1, \dots, \delta_u \in \mathbb{N}$ with
$$A_i(n) = \mathrm{ev}(a_i, n) \quad \forall n \geq \delta_i. \tag{12}$$
2. Set $W = \{T_1, \dots, T_e\}$ with $T_i := \mathsf{expr}(t_i) \in \Sigma\Pi(\mathbb{G})$ for $1 \leq i \leq e$.
3. Set $B_i := \mathsf{expr}(a_i) \in \mathsf{SumProd}(W, \mathbb{G})$ for $1 \leq i \leq u$.
4. Return W, $(B_1, \dots, B_u)$ and $(\delta_1, \dots, \delta_u)$.

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An Algorithm to solve Problem SigmaReduce

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Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

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Sigma computes

$$g(k) = (S_1(k) - 1)k.$$

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Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$

$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference ring for the summand

Consider a ring

$$\mathbb{A}$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

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Consider a ring

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$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(x) &= x + 1, & \mathcal{S}\, k &= k + 1, \\ \sigma(h) &= h + \frac{1}{x+1}, & \mathcal{S}\, S_1(k) &= S_1(k) + \frac{1}{k+1}. \end{aligned}$$

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Polynomial Solution: FIND

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h].$$

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$$\begin{aligned}&\big[\sigma(g_2\, h^2 + g_1 h + g_0)\big]\\ &\qquad\qquad -\big[g_2\, h^2 + g_1 h + g_0\big] = h\end{aligned}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

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g=hx-x

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Remarks. All results can be generalized to the following setting:

- the mixed multibasic hypergeometric case:
  $\mathbb{G} := \mathbb{K}(x, x_1, \ldots, x_v)$ with $\mathbb{K} = K(q_1, \ldots, q_v)$ For $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x, x_1, \ldots, x_v]$ where $q \neq 0$ and p, q being coprime we define

$$\operatorname{ev}(f, k) = \begin{cases} 0 & \text{if } q(k, q_1^k, \ldots, q_v^k) = 0 \\ \frac{p(k, q_1^k, \ldots, q_v^k)}{q(k, q_1^k, \ldots, q_v^k)} & \text{if } q(k, q_1^k, \ldots, q_v^k) \neq 0. \end{cases}$$

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- simple products: $\mathsf{Prod}^*(\mathbb{G})$ is the smallest set that contains 1 with:

1. If $r \in \mathcal{R}$ then $\mathsf{RPow}(r) \in \mathsf{Prod}^*(\mathbb{G})$.
2. If $f \in \mathbb{G}^*$, $l \in \mathbb{N}$ with $l \geq Z(f)$ then $\mathsf{Prod}(l, f) \in \mathsf{Prod}^*(\mathbb{G})$.
3. If $p, q \in \mathsf{Prod}^*(\mathbb{G})$ then $p \odot q \in \mathsf{Prod}^*(\mathbb{G})$.
4. If $p \in \mathsf{Prod}^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p^{\owedge} z \in \mathsf{Prod}^*(\mathbb{G})$.

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$$\mathrm{ev}(f, k) = \begin{cases} 0 & \text{if } q(k, q_1^k, \dots, q_v^k) = 0 \\ \frac{p(k, q_1^k, \dots, q_v^k)}{q(k, q_1^k, \dots, q_v^k)} & \text{if } q(k, q_1^k, \dots, q_v^k) \neq 0. \end{cases}$$

- nested products: $\mathrm{Prod}^*(\mathbb{G})$ is the smallest set that contains 1 with:

1. If $r \in \mathcal{R}$ then $\mathrm{RPow}(r) \in \mathrm{Prod}^*(\mathbb{G})$.
2. If $p \in \mathrm{Prod}^*(\mathbb{G})$, $f \in \mathbb{G}^*$, $l \in \mathbb{N}$ with $l \geq Z(f)$ then $\mathrm{Prod}(l, f \odot p) \in \mathrm{Prod}^*(\mathbb{G})$.
3. If $p, q \in \mathrm{Prod}^*(\mathbb{G})$ then $p \odot q \in \mathrm{Prod}^*(\mathbb{G})$.
4. If $p \in \mathrm{Prod}^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p^{\textcircled{\wedge}} z \in \mathrm{Prod}^*(\mathbb{G})$.

Remarks. All results can be generalized to the following setting:

- the mixed multibasic hypergeometric case:
  $\mathbb{G} := \mathbb{K}(x, x_1, \dots, x_v)$ with $\mathbb{K} = K(q_1, \dots, q_v)$ For $f = \frac{p}{q} \in \mathbb{G}$ with $p, q \in \mathbb{K}[x, x_1, \dots, x_v]$ where $q \neq 0$ and p, q being coprime we define

$$\operatorname{ev}(f, k) = \begin{cases} 0 & \text{if } q(k, q_1^k, \dots, q_v^k) = 0 \\ \frac{p(k, q_1^k, \dots, q_v^k)}{q(k, q_1^k, \dots, q_v^k)} & \text{if } q(k, q_1^k, \dots, q_v^k) \neq 0. \end{cases}$$

- nested products: $\mathsf{Prod}^*(\mathbb{G})$ is the smallest set that contains 1 with:

1. If $r \in \mathcal{R}$ then $\mathsf{RPow}(r) \in \mathsf{Prod}^*(\mathbb{G})$.
2. If $p \in \mathsf{Prod}^*(\mathbb{G})$, $f \in \mathbb{G}^*$, $l \in \mathbb{N}$ with $l \geq Z(f)$ then $\mathsf{Prod}(l, f \odot p) \in \mathsf{Prod}^*(\mathbb{G})$.
3. If $p, q \in \mathsf{Prod}^*(\mathbb{G})$ then $p \odot q \in \mathsf{Prod}^*(\mathbb{G})$.
4. If $p \in \mathsf{Prod}^*(\mathbb{G})$ and $z \in \mathbb{Z} \setminus \{0\}$ then $p^{\textcircled{\wedge}} z \in \mathsf{Prod}^*(\mathbb{G})$.

For further details see

Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation. To appear in: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computuation, 2021. Springer, arXiv:2102.01471 [cs.SC]

General picture:

Part 1: Symbolic summation (a short introduction)

Part 2: Modeling of sequences with a term algebra (user interface)

Part 3: Modeling of sequences in difference rings (computer algebra)

Part 4: Construction of appropriate difference rings (advanced CA)

Part 5: Applications

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}$$
$$\underbrace{+\frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \quad (=H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

$\uparrow$ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)})$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[8]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= $$\mathbf{mySum} = \sum_{\mathbf{j=0}}^{\mathbf{a}} \Big(\frac{\mathbf{(2j+k+n+2)j!k!(j+k+n)!}}{\mathbf{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}} + \frac{\mathbf{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}}{\mathbf{(j+k+1)!(j+n+1)!(k+n+1)!}} \Big);$$

In[8]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[10]:= $\mathbf{res} = \mathbf{SigmaReduce[mySum]}$

Out[10]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S_1[a]-S_1[a+k]-S_1[a+n]+S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[9]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1[j]+S_1[j+k]+S_1[j+n]-S_1[j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[10]:= **res = SigmaReduce[mySum]**

Out[10]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S_1[a]-S_1[a+k]-S_1[a+n]+S_1[a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[11]:= **SigmaLimit[res, {n}, a]**

Out[11]= $$\frac{1}{n!}\frac{S_1[k]+S_1[n]-S_1[k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \qquad \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Telescoping

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no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

no solution ☹

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Zeilberger's creative telescoping paradigm

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$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

$\boxed{\text{Sigma } \textit{computes}:}$ $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

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Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

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Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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Summing this equation over k from 1 to a gives:

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Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\begin{array}{ccc} || & & || \\ \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} & & -\,n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1) \\ +\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} & & \end{array}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} \in \boxed{\left\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R}\right\}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[12]:= $$\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

In[12]:= $\mathbf{mySum = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};}$

Compute a recurrence

In[13]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[13]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[12]:= $$\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$$

Compute a recurrence

In[13]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[13]= $$-n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$$

In[14]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[14]= $$-\mathtt{n}\mathsf{SUM}[\mathtt{n}]+(1+\mathtt{n})(2+\mathtt{n})\mathsf{SUM}[\mathtt{n}+1] == \frac{(\mathtt{n}+1)\mathtt{S}[1,\mathtt{n}]+1}{(\mathtt{n}+1)^3}$$

In[12]:= $\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{a}} \frac{\mathbf{S[1,k] + S[1,n] - S[1,k+n]}}{\mathbf{kn(k+n+1)}};$

Compute a recurrence

In[13]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[13]= $n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[14]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[14]= $-n\mathsf{SUM}[n] + (1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[15]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[15]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

In[12]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[13]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[13]= $n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(a+1)(S[1,a]+S[1,n]-S[1,a+n])}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[14]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[14]= $-n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[15]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[n]]}$

Out[15]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}\}\}$

Combine the solutions

In[16]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[16]= $\frac{S[1,n]^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(n,k,j)}\Big)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

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2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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FIND all solutions expressible by indefinite nested products/sums
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3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

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$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+ \frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+ \frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}]}$

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Sigma - A summation package by Carsten Schneider © RISC-Linz

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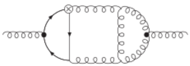
In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}]}$

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

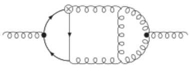
Application: The simplification of Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

Evaluation of Feynman Integrals



Behavior of particles

$$\longrightarrow \quad \int \Phi(N, \epsilon, x)dx$$

Feynman integrals

$$\int_0^1 x\,dx = \text{?}$$

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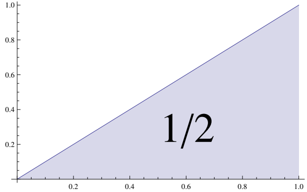
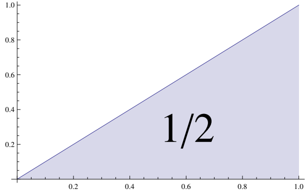
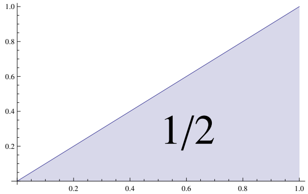


$$\int_0^1 x^1 \, dx =$$



$$\int_0^1 x^2 \, dx = \text{?}$$

$$\int_0^1 x^1\,dx =$$



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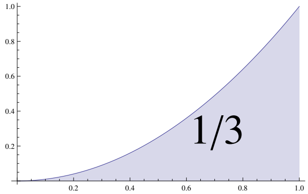

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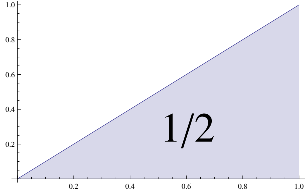
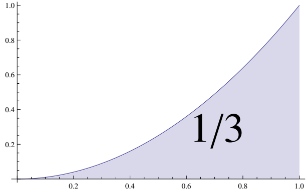
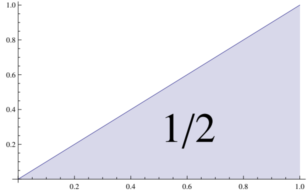


$$\int_0^1 x^2\,dx =$$

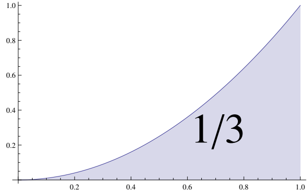


$$\int_0^1 x^3\,dx = \text{?}$$

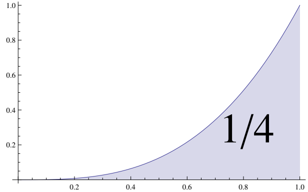
$$\int_0^1 x^1\,dx =$$



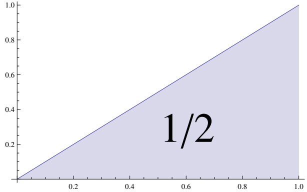
$$\int_0^1 x^2\,dx =$$



$$\int_0^1 x^3\,dx =$$



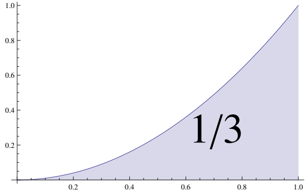
$$\int_0^1 x^1\,dx =$$



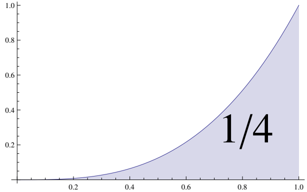
$$\boxed{\int_0^1 x^N\,dx = \frac{1}{N+1}}$$

für $N = 1, 2, 3, 4, \ldots$

$$\int_0^1 x^2\,dx =$$



$$\int_0^1 x^3\,dx =$$



Feynman integrals

$$\int_0^1 x^N \, dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N(1+x)^N}{(1-x)^{1+\varepsilon}}\,dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \dots dx_1 \, dx_2 \, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

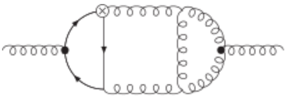
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\|$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

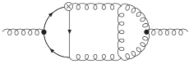
$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Evaluation of Feynman Integrals

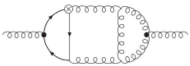


Behavior of particles

$$\longrightarrow \quad \int \Phi(N, \epsilon, x)dx$$

Feynman integrals

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N, \epsilon, x)dx$$

Feynman integrals

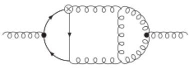
DESY

(J. Blümlein)

$$\sum f(N, \epsilon, k)$$

complicated multi-sums

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

DESY
(J. Blümlein)

$$\sum f(N,\epsilon,k)$$

complicated
multi-sums

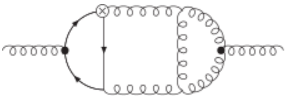
RISC
(Sigma-package)

expression in
special functions

Example 1:

massive 3-loop ladder integrals

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

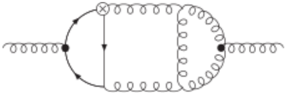
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-ep/2}$$

$$\Bigg[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$

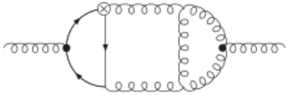
$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k\Bigg]$$

$$\times (1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times \left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$$
\boxed{F_0(N)} =
$$

$$
\begin{aligned}
&\frac{7}{12}S_1(N)^4+\frac{(17N+5)S_1(N)^3}{3N(N+1)}+\Big(\frac{35N^2-2N-5}{2N^2(N+1)^2}+\frac{13S_2(N)}{2}+\frac{5(-1)^N}{2N^2}\Big)S_1(N)^2\\
&+\Big(-\frac{4(13N+5)}{N^2(N+1)^2}+\Big(\frac{4(-1)^N(2N+1)}{N(N+1)}-\frac{13}{N}\Big)S_2(N)+\Big(\frac{29}{3}-(-1)^N\Big)S_3(N)\\
&+\big(2+2(-1)^N\big)S_{2,1}(N)-28S_{-2,1}(N)+\frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N)+\Big(\frac{3}{4}+(-1)^N\Big)S_2(N)^2\\
&-2(-1)^NS_{-2}(N)^2+S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)}+\big(26+4(-1)^N\big)S_1(N)+\frac{4(-1)^N}{N+1}\Big)\\
&+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)}-\frac{5}{2N^2}\Big)S_2(N)+S_{-2}(N)\Big(10S_1(N)^2+\Big(\frac{8(-1)^N(2N+1)}{N(N+1)}\\
&+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N)+\frac{8(-1)^N(3N+1)}{N(N+1)^2}+\big(-22+6(-1)^N\big)S_2(N)-\frac{16}{N(N+1)}\Big)\\
&+\Big(\frac{(-1)^N(9N+5)}{N(N+1)}-\frac{29}{3N}\Big)S_3(N)+\Big(\frac{19}{2}-2(-1)^N\Big)S_4(N)+\big(-6+5(-1)^N\big)S_{-4}(N)\\
&+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)}-\frac{2}{N}\Big)S_{2,1}(N)+\big(20+2(-1)^N\big)S_{2,-2}(N)+\big(-17+13(-1)^N\big)S_{3,1}(N)\\
&-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N)-\big(24+4(-1)^N\big)S_{-3,1}(N)+\big(3-5(-1)^N\big)S_{2,1,1}(N)\\
&+32S_{-2,1,1}(N)+\left(\frac{3}{2}S_1(N)^2-\frac{3S_1(N)}{N}+\frac{3}{2}(-1)^NS_{-2}(N)\right)\zeta(2)
\end{aligned}
$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 - \frac{(17N+5)S_1(N)^3}{\text{[illegible]}} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$+ \Big(- \text{[illegible]} \frac{\text{[illegible]}1)^N(2N+1)}{\text{[illegible]}N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$\boxed{F_0(N)} =$

$$\frac{7}{12}S_1(N)^4 [illegible] \frac{(17N+5)S_1(N)^3}{[illegible]} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+ \Big(- [illegible] 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{[illegible]} - (-1)^N\Big)S_3(N)$$
$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1 [illegible]} [illegible] {}^N\big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4([illegible] \frac{[illegible])^N}{N+1}\Big)$$
$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 - \frac{(17N+5)S_1(N)^3}{\cdots} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+\Big(-\cdots\frac{\cdots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{\cdots} - (-1)^N\Big)S_3(N)$$

$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N \cdots}{N^2(N+1\cdots)} \cdots {}^N\big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(\cdots\frac{\cdots)^N}{N+1}\Big)$$

$$+\Big(\frac{(-1)^N \cdots}{2N^2}\cdots\Big)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-\cdots}{N(N+\cdots}\cdots(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N\cdots}{N(\cdots}\cdots(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-\cdots}{N(N+1)}\cdots\frac{\cdots}{N}\Big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

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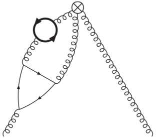
$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

Example 2:

2-mass 3-loop Feynman integrals

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



All diagrams are produced with axodraw (J. Vermaseren).

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

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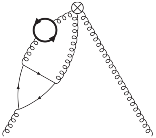

$$\xrightarrow[\text{and } {}_pF_q\text{-technologies}]{\text{Mellin-Barnes-}}$$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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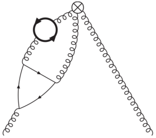

$$\xrightarrow[]{\text{Mellin-Barnes- and } {}_pF_q\text{-technologies}}$$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

Typical triple sum:

$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i} \frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon} \times 2^{-2+\varepsilon} e^{-\frac{3\varepsilon\gamma}{2}} \eta^k \times$$

$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

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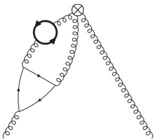

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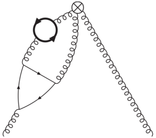
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

6 hours for this sum

$\sim$ 10 years of calculation time for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow[\text{and } {}_pF_q\text{-technologies}]{\text{Mellin-Barnes-}}$

expression (95 MB) with

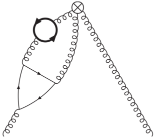
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow{\text{Mellin-Barnes- and } {}_pF_q\text{-technologies}}$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m

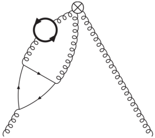
Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation		number of indef. sums
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s	(2.1 days)	1188
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s	(11.4 days)	747
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s	(6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s	(8.0 days)	44
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s	(3.5 days)	1933
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s	(8.2 days)	621
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s	(5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s	(2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



$\xrightarrow{\text{Mellin-Barnes- and } {}_pF_q\text{-technologies}}$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB)
consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

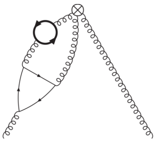
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left(\sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \displaystyle\sum_{j=1}^{i} \frac{\displaystyle\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

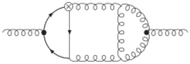
$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

$\longleftarrow$ Sigma.m (32 days)

expression (8.3 MB) consisting of 74 indefinite sums

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

DESY
(J. Blümlein)

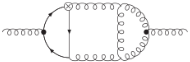
$$\sum f(N,\epsilon,k)$$

complicated
multi-sums

RISC
(Sigma-package)

expression in
special functions

Evaluation of Feynman Integrals



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals

LHC at CERN

DESY

(J. Blümlein)

applicable

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

RISC

(Sigma-package)

expression in special functions