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Formal Modeling
Rigid Body Dynamics

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Abstract

These lecture notes contain the material on balancing problems. In these notes, the equations for the motions of a rigid body are derived, the free motion of a rigid body is described, and the balancing problem for certain mechanisms is explained.

1 The Motion Equations

We define angular velocity, angular momentum, and kinetic energy of a rigid body moving in 3-space, and we derive the physical laws relating these quantities.
A rigid body is moving in 3-space is modeled as a finite set of points, where each point has a particular mass; the distance between two points stays fixed throughout the motion.
Let $n \in \mathbb{N}$ be the number of points and $m_1, \ldots, m_n$ be the masses; clearly, $m_1, \ldots, m_n$ are positive real numbers. The *total mass* is $M := \sum_{i=1}^{n} m_i$. Let $p_1, \ldots, p_n \in \mathbb{R}^3$ are the positions of the n points, then the *center of mass* is the point

$$p_c := \frac{\sum_{i=1}^{n} m_i p_i}{M}.$$

In this note, we assume that $p_c = 0$ (this simplifies some of the computations below). Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ be a map that preserves the distance between points. Then σ can be written as

$$\sigma : x \mapsto Rx + a,$$

where $R \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix and $a \in \mathbb{R}^3$ is a vector. Orthogonal matrices have determinant ± 1, and in the case of the motion of a rigid body, the determinant of the matrix in the map transforming the points at time t_0 to the points at time t_1 is always $+1$ by continuity. All maps $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Rx + a$ such that R is orthogonal and $\det(R) = 1$ are called *displacements*. The set of displacements form a group, which we denote by G.

A motion is now a function $f : T \to G$, where $T \subset \mathbb{R}$ is an interval containing 0, which is two times continuously differentiable. The value of 0 is equal to the identity. We assume that $f(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the given by $x \mapsto R(t)x + a(t)$, where $a : T \to \mathbb{R}^3$ is a vector-valued function and $R : T \to \mathrm{SO}(3)$ is a function from T to the group of orthogonal matrices with determinant 1. The derivative of a at 0 is denoted by V – it is the velocity of the center of mass. The second derivative of a at 0 is denoted by A – it is the acceleration of the center of mass. The derivative of R at 0 is skew symmetric, because

$$0 = \frac{dR(t)R(t)^\dagger}{dt}|_{t=0} = R(0)R'(0)^\dagger + R'(0)R(0)^\dagger = R'(0)^\dagger + R'(0).$$

For any skew symmetric matrix Γ, there is a unique vector γ such that $\Gamma x = \gamma \times x$ for all $x \in \mathbb{R}^3$; if $\Gamma = R'(0)$, then the vector γ is called the *angular velocity* and denoted by ω. The velocity of the i-th point at $t = 0$ is equal to $R'(0)p_i + a'(0) = \omega \times p_i + V$. The *total momentum* of the rigid body is

$$P := \sum_{i=1}^{n} m_i(\omega \times p_i + V) = \omega \times \sum_{i=1}^{n} m_i p_i + MV = MV,$$

which is equal to the momentum of a point mass M at the center of mass. If the sum of all forces exerted to the the i points (also called the *total force*) is zero, then the momentum is constant by Newtons second axiom: the center of mass travels with uniform speed in a constant direction.

In order to define the *angular momentum*, we have to choose a reference point. For simplicity, we choose the origin, which is already the center of mass at time 0. The angular momentum of the i-th point is defined as product the cross product of the position vector with respect to the reference point and the momentum vector. The total angular momentum is

$$L := \sum_{i=1}^{n} m_i p_i \times (\omega \times p_i + V) = \sum_{i=1}^{n} m_i p_i \times (\omega \times p_i) + \left(\sum_{i=1}^{n} m_i p_i\right) \times V.$$

The second summand is zero. The first summand is linear in ω and can therefore be written as $I\omega$ for a suitable matrix $I \subset \mathbb{R}^{3\times 3}$ (please note that this time I is not the unit matrix). The matrix I is called the *moment of inertia*. If $p_i =: (x_i, y_i, z_i)$, then the contribution of the i-th point to the moment of inertia is $m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix}$. This matrix is symmetric and positive semidefinite, hence it follows that I is also symmetric and positive semidefinite.

If the reference point is not equal to the center of mass, then the second summand in the line defining L is not zero, but is equal to the cross product of the position vector of the center of mass and the momentum P.

According to Newton's axioms, when the rigid body moves then there must be some forces exerted to the mass points that ensure that the distance are preserved (for instances by rods or cables). For any two distinct indices i, j, there is a force $F_{i,j}$ that pulls p_i into the direction of p_j. By Newton's third axiom, $F_{ji} = F_{ij}$. If there are no other forces, then the derivative of the angular momentum at 0 is

$$\sum_{i \neq j}^{n} p_i \times F_{ij} = \sum_{i<j}^{n} (p_i - p_j) \times F_{ij} = 0.$$

Therefore the angular momentum is constant if there are no external forces and if the internal forces between two points have the same direction as the difference vector. If there are external forces, then the derivative if the angular momentum is called the *torque* and denoted by T.

The kinetic energy of the i-th point is equal to mass times square of velocity divided by 2. The *total energy* is then

$$E := \frac{1}{2}\sum_{i=1}^{n} m_i ||\omega \times p_i + V||^2 = \frac{1}{2}\sum_{i=1}^{n} m_i ||\omega \times p_i||^2 + \sum_{i=1}^{n} m_i \langle \omega \times p_i \mid V \rangle + \frac{1}{2}\sum_{i=1}^{n} m_i ||V||^2.$$

The second summand is equal to $\langle \omega \times (\sum_{i=1}^{n} m_i p_i) \mid V \rangle = 0$. The third summand is equal to the kinetic energy of a point mass concentrated in the center of mass. For the first summand, which is called the *rotational energy*, we use the general formula $||a \times b||^2 = \langle a \mid b \times (a \times b) \rangle$ for $a, b \in \mathbb{R}^3$ (in our case, $a = \omega$ and $b = p_i$):

$$\frac{1}{2}\sum_{i=1}^{n} m_i ||\omega \times p_i + V||^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \langle \omega \mid p_i \times (\omega \times p_i) \rangle = \frac{1}{2}\langle \omega \mid \sum_{i=1}^{n} m_i p_i \times (\omega \times p_i) \rangle$$
$$= \frac{1}{2}\langle \omega \mid L \rangle = \frac{1}{2}\langle \omega \mid I\omega \rangle$$

The total energy is preserved when there are no external forces. Newton's three axioms do not exclude internal forces that are not gradients, and such forces would in general change the total energy. However, the conservation of energy may be considered as an additional axiom for mechanics, motivated by the impossibility of a device that constantly produces energy.

Here is the summary of the laws of motion of a rigid body whose center of mass is at the reference point:

$$P = MV, L = I\omega, E = \frac{MV^2}{2} + \frac{\langle \omega, I\omega \rangle}{2}.$$

If two rigid bodies have the same center of mass and the same moment of intertia, then they move exactly in the same way, provided that the external force and torque is the same in both cases.

2 The Free Motion

If a rigid body moves without external force or torque, its angular velocity is in general not constant. We derive equations for the geometric locus of the angular velocity vector in the fixed and in the moving coordinate system.

Assume that a rigid body moves without external force or torque. Then the center of mass oves uniformly with constant speed. Moving the reference frame with the same velocity compensates that and does not affect the motion equations, hence we may assume, without loss of generality, that the center of mass remains fixed at the origin. The motion is descrived by a function $R : T \to \mathbb{R}^{n\times n}$ as above, i.e., T is an interval containing 0 and R is two times continuously differentiable, $R(0)$ is the identity, and $R(t)$ is orthogonal for all $t \in T$. The momentum P is zero. In order to define angular momentum at time t, we need a skew symmetric matrix related to $R'(t)$. The matrix $R'(t)$ is in general not skew symmetric, but the matrices $R(t)^\dagger R'(t)$ and $R'(t)R(t)^\dagger$ are both skew symmetric. Hence we get two vectors of angular velocity implicitly defined by the two equations

$$\forall p \in \mathbb{R}^3\ \forall t \in T : \omega_f(t) \times R(t)p = R'(t)p,\ \ R(t)(\omega_m(t) \times p) = R'(t)p.$$

The two angular velocity vectors are related by the equation $\omega_f(t) = R(t)\omega_m(t)$. The first one, $\omega_f(t)$, expresses angular velocity in a fixed coordinate frame, and the second, $\omega_m(t)$, expresses it in a coordinate frame that is attached to the rigid body and moves with it.

The moment of inertia I is constant in the moving coordinate frame, because the vectors p_i are constant in the moving frame. This gives the formula $L_m(t) = I\omega_m(t)$ for the angular momentum in the moving frame: the vector itself is constant, but when the coordinate frame changes then also the coordinates change. On the other hand, the vector $L_f = R(t)I\omega_f(t) = R(t)IR^(t)^\dagger\omega_f(t)$ is fixed, since this is the angular momentum in the fixed frame.

The total energy can be computed in both frames. For the fixed frame, we get

$$E = \frac{1}{2}\langle \omega_f(t) \mid L_f\rangle.$$

This is a linear equation for $\omega_f(t)$, which shows that ω_f has values in a plane. However, it is almost impossible to obtain a second equation for $\omega_f(t)$, let alone a closed formular for ω_f.

In the moving frame, we get the equation

$$E = \frac{1}{2}\langle \omega_m(t) \mid I\omega_m(t)\rangle,$$

which is a quadratic equation for ω_m. A second quadratic equation is obtained using the angular momentum. It is not fixed in the moving frame, but its length is independent of the choice of the frame:

$$||I\omega_m(t)||^2 = ||L_f||^2.$$

In general (if the inertial matrix is not a multiple of the unit matrix), these two equations are linearly independent, and $\omega_f(t)$ moves along the common zero set of these two quadratic equations.
Here is a beautiful animation `http://www.ialms.net/sim/3d-rigid-body-simulation/` of the free motion of a rigid body displaying also angular momentum, angular velocity, and the loci of angular velocity in fixed and moving coordinates.

3 Balancing of Mechanisms

The material below was not yet treated in the lecture.
A *mechanism* consists of several rigid bodies called *links* that may or may not be connected by *revolute joints* (there exist also other types of joints, but in this note, only revolute joints appear). All links may move in 3-space. If two links are connected by a joint, then the relative motion of the second with respect to the first is restricted by a line: in both links, all points on the line are fixed. If we fix the first link, then the second link can only rotate around the line, which is also called the *joint axes*.
The easiest possible nontrivial mechanism consists of two links connected by a joint. Slightly more general, let $n \in \mathbb{N}$. A linkage consisting of $n+1$ links $L_0, \ldots, L_n$ and joints J_r connecting L_{r-1} and L_r ifor $r = 1, \ldots, n$ is called an nR-chain. (The "R" specifies the type of joint, which in our case is always revolute.) For $r = 1, \ldots, n$, the relative position of L_r with respect to L_{r-1} can be specified by a single *joint parameter*, an angle $[0, 2\pi)$. The position of L_n with respect to L_0 depends on all joint parameters and can be computed as a composition of n rotations. 6R-chains are frequently used in robotics, especially in car manufactury: the joint parameters are controlled by programs such that the link L_6 – which is called *hand* in robotics – performs a prescribed motion relative to the fixed base L_0 (containing the partially constructed car).
For a fixed nR-chain, the *balancing problem* is to add masses to the individual links such that for all possible motions of the mechanism, the position of the center of mass of the whole mechanism is fixed with respect to the link L_0. If an nR-chain is balanced, then every configuration of the mechanism, i.e. every displacement of the links that is consistent with the joint restrictions, is in an indifferent equilibrium and does not move away by gravity forces.
The data needed to solve the balancing problem are the following: for each link, we need the mass, the position of the rotation axis (two for the middle links, one for the base and one for the hand) and the position of the center of mass. In addition, we need two *assembling points* on each axes. When the mechanism is assembled from its individual links, the lines representing the same joint must be equal, and also corresponding assembling points must be equal. (Without specifying the assembling points, it would not be clear which points of an axes on link i would be equal to which points on the same axes on link L_{i+1}.) It should be noted that the distance between corresponding assembling points must be equal, otherwise it is not possible to assemble the linkage. Also, it should it be noted that the different links have in general different coordinate systems. The relative position of the

two coordinate system changes when the mechanism moves.
The balancing problem can be recursively solved in the following way: first, add a point mass to the link L_n that ensures that the center of mass of L_n is located in the joint axis of J_n. Then replace the link L_n by a point with all its mass concentrated in the center of mass. Consider this point mass as attached to L_{n-1} and remove the last link L_n. If $n = 1$, then the mechanism is now balanced; otherwise, balance the $(n-1)$R chain consisting of the links $L_0, \dots, L_{n-1}$ (where the last link has been changed by the previous procedure).
Every solution of the balancing problem of an nR-chain can be obtained by the method above: if the center of mass of L_n would not lie on the joint axis of J_n, then a rotation of L_n around L_{n-1}, with $L_0, \dots, L_{n-1}$ fixed, would be a possible motion that changes the position of the whole center of mass. Hence the center of mass of L_n of a balanced nR-chain *has to lie* on the joint axis. But then, the balancing problem for the nR-chain is equivalent to the balancing problem of the $(n-1)$R-chain.
If we apply the construction to a mechanism that is already balanced, then no masses are added. This observation allows to *decide* algorithmically if a given nR chain is balanced: just run the balancing algorithm and check if masses are added or not.
We now define another type of mechanism, the nR-loop. It consists of n links $L_1, \dots, L_n$ that are cyclically connected by joints: J_1 connects L_n and L_1, J_2 connects L_1 and L_2 etc. It can also be constructed from an nR-chain by fixing the relative position of L_n with respect to L_0 in an nR-chain (e.g. when the robot hand firmly grips a part of the car).
The balancing problem for nR-loops is defined similar as for the nR-chain: to add masses to the links such that the center of mass is fixed relative to L_1 for all possible motions in the configuration space. By temporily breaking a joint, say J_k, we can construct soluton by balancing the $(k-2)$-chain consisting of links $L_1, L_2, \dots, L_{k-1}$ and the the $(n-k+1)$-chain consisting of links $L_1, L_n, L_{n-1}, \dots, L_k$. So the balancing problem for loops is always solvable.
However, in general, there are more solutions than those obtained by breaking the loop into two chains. Also, if we start with an algorithm that is already balanced, the two chains are in general not balanced, so our construction is adding masses. We cannot use the construction to decide if a given algorithm is already balanced.
Another approach is to try to distribute the masses to the joint axes. For the moment, let us consider just a single link. Let us also assume that the tw axes are skew. Here is a useful geometric lemma.

Lemma 3.1. *Let L_1, L_2 be skew lines. Let p be a point which is not in L_1 nor in L_2, such that the line parallel to L_1 through p does not meet L_2 and the line parallel to L_2 through p does not meet L_1. Then there is a unique line through p meeting both L_1 and L_2.*

The proof is left as an exercise. – If the conditions in the Lemma are fulfilled for the two rotation axes and the center of mass, then the center of mass p_c can (uniquely!) be written in the form

$$p_c = \frac{m_1 p_1 + m_2 p_2}{M}$$

with m_1, m_2 are real summing to the mass M of the link, and p_1 is on the first axis, and p_2 is on the second axis. Note that m_1, m_2 could be negative!
It is clear that the mechanism is balanced if the contributions by mass distribution to a any joint axis that is actually moving, that is, all joint axis except the two attached to the base link, compensate each other. For any such axes, there are two contributions from the two attache links. "Compensation of each other" means that the two points are equal and the two virtual masses (i.e., the m_1 and m_2 in the above formula) sum up to zero.
The above method gives a sufficient condition for balancing. For most known nR-linkages, the condition "after distributing to the axes, all contributions to moving joint axes componensate each other" is also necessary.